Problem 14

In each of Problems 13 and 14, solve the given initial value problem and determine at least approximately where the solution is valid.

$$(9x^2 + y - 1) - (4y - x)y' = 0, \qquad y(1) = 0$$

Solution

Distribute the minus sign in the second term.

$$(9x^2 + y - 1) + (x - 4y)y' = 0$$

The ODE is exact because

$$\frac{\partial}{\partial y}(9x^2 + y - 1) = \frac{\partial}{\partial x}(x - 4y) = 1.$$

That means there exists a potential function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = 9x^2 + y - 1 \tag{1}$$

$$\frac{\partial \psi}{\partial y} = x - 4y. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = 3x^3 + xy - x + f(y)$$

Here f is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = x + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = -4y \quad \rightarrow \quad f(y) = -2y^2.$$

As a result, a potential function is

$$\psi(x, y) = 3x^3 + xy - x - 2y^2.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

$$3x^3 + xy - x - 2y^2 = C$$

Apply the condition $y(1) = 0$ now to determine C.

$$3(1)^3 + (1)(0) - (1) - 2(0)^2 = C \quad \rightarrow \quad C = 2$$

Therefore,

$$3x^3 + xy - x - 2y^2 = 2.$$

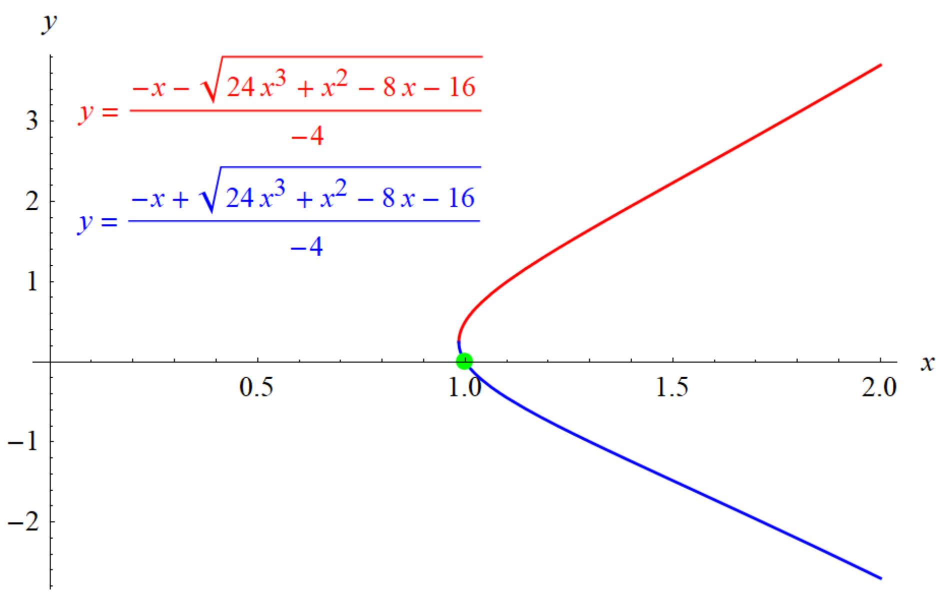
Solve for y using the quadratic formula.

$$-2y^2 + xy - x + 3x^3 - 2 = 0$$

$$y = \frac{-x \pm \sqrt{x^2 - 4(-2)(-x + 3x^3 - 2)}}{2(-2)}$$

$$y = \frac{-x \pm \sqrt{24x^3 + x^2 - 8x - 16}}{-4}$$

Below is a plot of these two functions, and the condition $y(1) = 0$ is represented by a green dot.



The blue curve is in contact with the dot, so the solution to the initial value problem is

$$y = \frac{-x + \sqrt{24x^3 + x^2 - 8x - 16}}{-4},$$

which is valid roughly when

$$x \gtrsim 0.98.$$