Problem 18

Show that any separable equation

$$M(x) + N(y)y' = 0$$

is also exact.

Solution

This separable equation is exact because

$$\frac{\partial}{\partial y}[M(x)] = \frac{\partial}{\partial x}[N(y)] = 0,$$

and there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = M(x) \tag{1}$$

$$\frac{\partial \psi}{\partial y} = N(y). \tag{2}$$

To find it, integrate both sides of equation (1) partially with respect to x.

$$\psi(x, y) = \int^x M(s)\, ds + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = N(y) \quad \rightarrow \quad f(y) = \int^y N(t)\, dt.$$

Therefore, a potential function is

$$\psi(x, y) = \int^x M(s)\, ds + \int^y N(t)\, dt.$$