Problem 23

Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)\, dy.$$

Solution

The ODE in question is

$$M(x, y) + N(x, y)y' = 0.$$

In general, this will not be an exact equation, so we will multiply both sides by the integrating factor $\mu = \mu(x, y)$ to make it so.

$$\mu M(x, y) + \mu N(x, y)y' = 0$$

Now that it is exact,

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N).$$

Expand both sides.

$$\frac{\partial \mu}{\partial y} M + \mu M_y = \frac{\partial \mu}{\partial x} N + \mu N_x$$

Assume now that μ is dependent on y only: $\mu = \mu(y)$.

$$\frac{d\mu}{dy} M + \mu M_y = \mu N_x$$

Solve for $d\mu/dy$.

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$$

Divide both sides by μ.

$$\frac{\frac{d\mu}{dy}}{\mu} = \frac{N_x - M_y}{M}$$

The left side can be written as the derivative of a logarithm by the chain rule, and the right side is $Q(y)$.

$$\frac{d}{dy} \ln \mu = Q(y)$$

Integrate both sides with respect to y.

$$\ln \mu = \int Q(y)\, dy$$

Therefore, exponentiating both sides, an integrating factor is

$$\mu(y) = \exp \int Q(y)\, dy.$$