Problem 13

In each of Problems 13 and 14, solve the given initial value problem and determine at least approximately where the solution is valid.

$$(2x - y) + (2y - x)y' = 0, \qquad y(1) = 3$$

Solution

The ODE is exact because

$$\frac{\partial}{\partial y}(2x - y) = \frac{\partial}{\partial x}(2y - x) = -1.$$

That means there exists a potential function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = 2x - y \tag{1}$$

$$\frac{\partial \psi}{\partial y} = 2y - x. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = x^2 - xy + f(y)$$

Here f is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = -x + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 2y \quad \rightarrow \quad f(y) = y^2.$$

As a result, a potential function is

$$\psi(x, y) = x^2 - xy + y^2.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x,y) = C$$

$$x^2 - xy + y^2 = C$$

Apply the condition $y(1) = 3$ now to determine C.

$$1^2 - (1)(3) + (3)^2 = C \quad \rightarrow \quad C = 7$$

Therefore,

$$x^2 - xy + y^2 = 7.$$

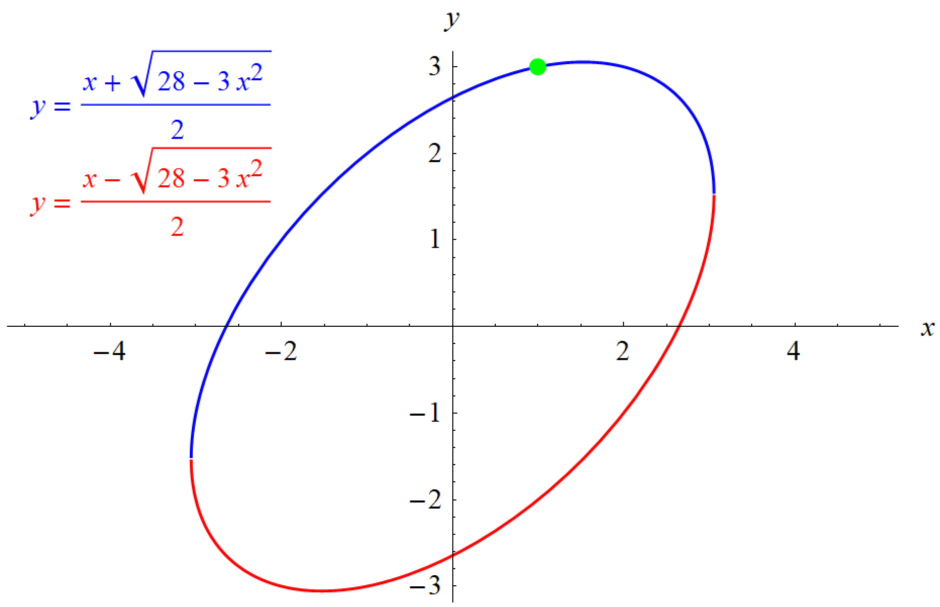
Solve for y using the quadratic formula.

$$y^2 - xy + (x^2 - 7) = 0$$

$$y = \frac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2}$$

$$y = \frac{x \pm \sqrt{28 - 3x^2}}{2}$$

Below is a plot of these two functions, and the condition $y(1) = 3$ is represented by a green dot.



The blue curve is in contact with the dot, so the solution to the initial value problem is

$$y(x) = \frac{x + \sqrt{28 - 3x^2}}{2},$$

which is valid when

$$28 - 3x^2 \ge 0$$

$$-\sqrt{\frac{28}{3}} \le x \le \sqrt{\frac{28}{3}}.$$