Problem 19

In each of Problems 19 through 22, show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

$$x^2y^3 + x(1+y^2)y' = 0, \qquad \mu(x,y) = 1/xy^3$$

Solution

The ODE is not exact because

$$\frac{\partial}{\partial y}(x^2y^3) = 3x^2y^2 \neq \frac{\partial}{\partial x}[x(1+y^2)] = 1+y^2.$$

However, by multiplying both sides of the ODE by $\mu(x,y)$, we get

$$x + \frac{1+y^2}{y^3}y' = 0,$$

which is exact because

$$\frac{\partial}{\partial y}(x) = \frac{\partial}{\partial x}\left(\frac{1+y^2}{y^3}\right) = 0.$$

That means there exists a potential function $\psi = \psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = x \tag{1}$$

$$\frac{\partial \psi}{\partial y} = \frac{1+y^2}{y^3}. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x,y) = \frac{x^2}{2} + f(y)$$

Here f is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x,y) = f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = \frac{1+y^2}{y^3} \quad \rightarrow \quad f(y) = -\frac{1}{2y^2} + \ln|y|.$$

As a result, a potential function is

$$\psi(x,y) = \frac{x^2}{2} - \frac{1}{2y^2} + \ln|y|.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

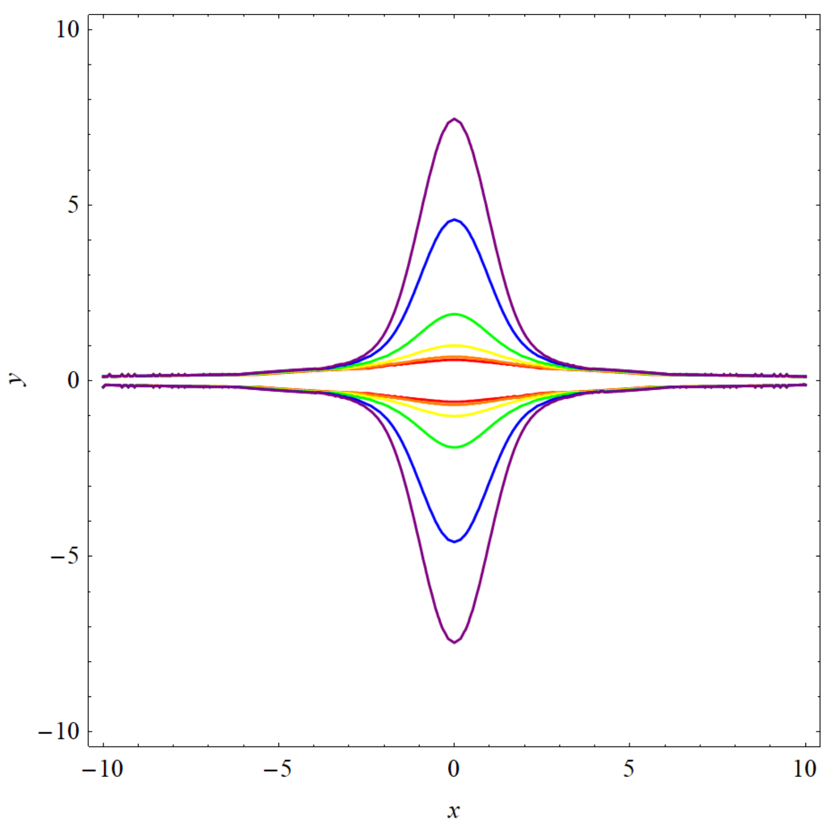
$$\frac{x^2}{2} - \frac{1}{2y^2} + \ln|y| = C$$

Multiply both sides by 2.

$$x^2 - \frac{1}{y^2} + 2\ln|y| = 2C$$

Therefore, using a new constant k for $2C$,

$$x^2 - \frac{1}{y^2} + \ln y^2 = k.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $k = -4$, $k = -3$, $k = -1$, $k = 1$, $k = 3$, and $k = 4$, respectively.