Problem 24

Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

Solution

The ODE in question is

$$M(x, y) + N(x, y)y' = 0.$$

In general, this will not be an exact equation, so we will multiply both sides by the integrating factor $\mu = \mu(x, y)$ to make it so.

$$\mu M(x, y) + \mu N(x, y)y' = 0$$

Now that it is exact,

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N).$$

Expand both sides.

$$\frac{\partial \mu}{\partial y}M + \mu M_y = \frac{\partial \mu}{\partial x}N + \mu N_x$$

Assume now that μ is dependent on xy only: $\mu = \mu(xy)$.

$$x\mu'(xy)M + \mu M_y = y\mu'(xy)N + \mu N_x$$

Solve for $\mu'(xy)$.

$$\mu'(xy)(xM - yN) = (N_x - M_y)\mu$$

$$\mu'(xy) = \frac{N_x - M_y}{xM - yN}\mu$$

Divide both sides by μ.

$$\frac{\mu'(xy)}{\mu} = \frac{N_x - M_y}{xM - yN}$$

Let $z = xy$. The left side can be written as the derivative of a logarithm by the chain rule, and the right side is $R(z)$.

$$\frac{d}{dz}\ln \mu = R(z)$$

Integrate both sides with respect to z.

$$\ln \mu = \int R(z)\, dz$$

Therefore, exponentiating both sides, an integrating factor is

$$\mu(xy) = \exp\left[\int^{xy} R(z)\, dz\right].$$