Problem 26

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

$$y' = e^{2x} + y - 1$$

Solution

Bring all terms to the left side.

$$(1 - e^{2x} - y) + y' = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(1 - e^{2x} - y) = -1 \neq \frac{\partial}{\partial x}(1) = 0.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(1 - e^{2x} - y)\mu + \mu y' = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(1 - e^{2x} - y)\mu] = \frac{\partial}{\partial x}(\mu).$$

Expand both sides.

$$-\mu + (1 - e^{2x} - y)\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$-\mu = \frac{d\mu}{dx}$$

Solve for μ.

$$\frac{d\mu}{dx} = -\mu$$

$$\frac{\frac{d\mu}{dx}}{\mu} = -1$$

$$\frac{d}{dx}\ln \mu = -1$$

$$\ln \mu = -x$$

As a result, an integrating factor is

$$\mu(x) = e^{-x}.$$

Multiply both sides of equation (1) by e^{-x}.

$$(e^{-x} - e^{x} - e^{-x}y) + e^{-x}y' = 0$$

Because it's exact, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = e^{-x} - e^{x} - e^{-x}y \tag{2}$$

$$\frac{\partial \psi}{\partial y} = e^{-x}. \tag{3}$$

Integrate both sides of equation (3) partially with respect to y to get ψ.

$$\psi(x,y) = e^{-x}y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x,y) = -e^{-x}y + f'(x)$$

Comparing this to equation (2), we see that

$$f'(x) = e^{-x} - e^{x} \quad \rightarrow \quad f(x) = -e^{-x} - e^{x}.$$

As a result, a potential function is

$$\psi(x,y) = e^{-x}y - e^{-x} - e^{x}.$$

Notice that by substituting equations (2) and (3), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x,y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

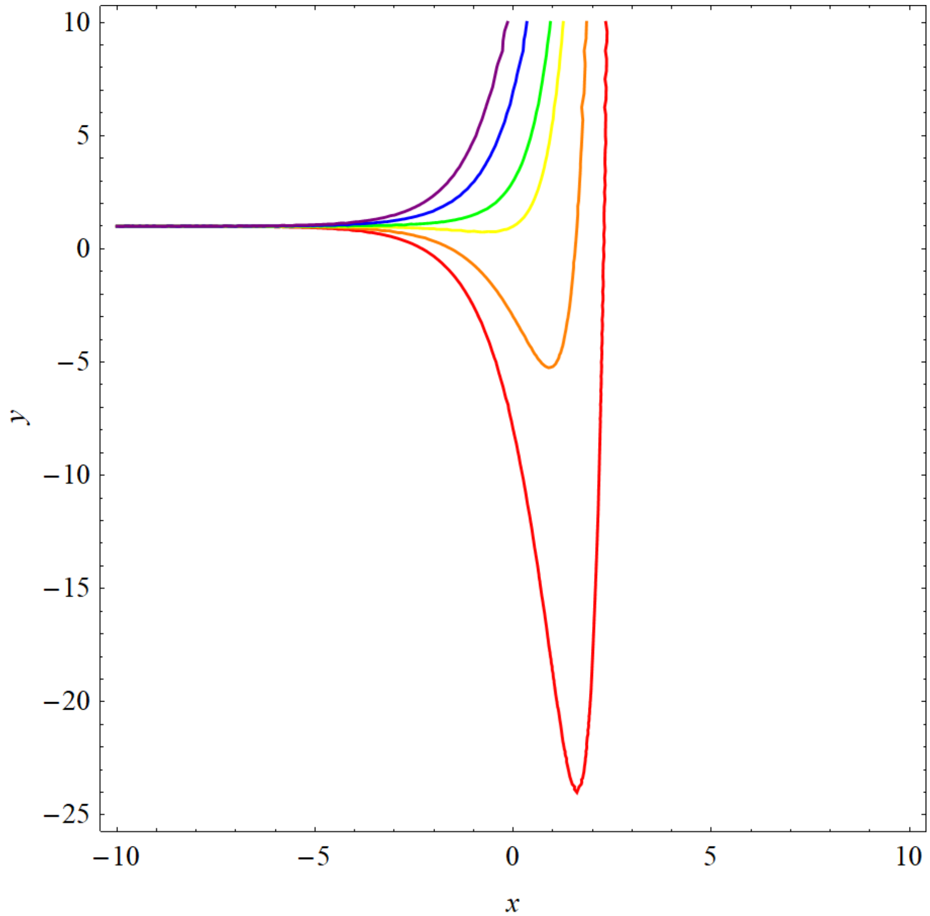
$$\psi(x,y) = C$$

Therefore,

$$e^{-x}y - e^{-x} - e^{x} = C,$$

or solving for y explicitly,

$$y(x) = e^{2x} + 1 + Ce^{x}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.