Problem 27

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

$$1 + (x/y - \sin y)y' = 0$$

Solution

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(1) = 0 \neq \frac{\partial}{\partial x}\left(\frac{x}{y} - \sin y\right) = \frac{1}{y}.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$\mu + \mu\left(\frac{x}{y} - \sin y\right)y' = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}(\mu) = \frac{\partial}{\partial x}\left[\mu\left(\frac{x}{y} - \sin y\right)\right].$$

Expand both sides.

$$\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}\left(\frac{x}{y} - \sin y\right) + \frac{\mu}{y}$$

Assume that μ is only dependent on y: $\mu = \mu(y)$.

$$\frac{d\mu}{dy} = \frac{\mu}{y}$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = \frac{dy}{y}$$

Integrate both sides.

$$\ln \mu = \ln y + C$$

Exponentiate both sides.

$$\mu = ye^C$$

Taking e^C to be 1, an integrating factor is

$$\mu = y.$$

Multiply both sides of the original ODE by y.

$$y + (x - y\sin y)y' = 0$$

Because it's exact, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = y \tag{1}$$

$$\frac{\partial \psi}{\partial y} = x - y\sin y. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = xy + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = x + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = -y \sin y \quad \rightarrow \quad f(y) = y \cos y - \sin y.$$

As a result, a potential function is

$$\psi(x, y) = xy + y \cos y - \sin y.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (3) becomes

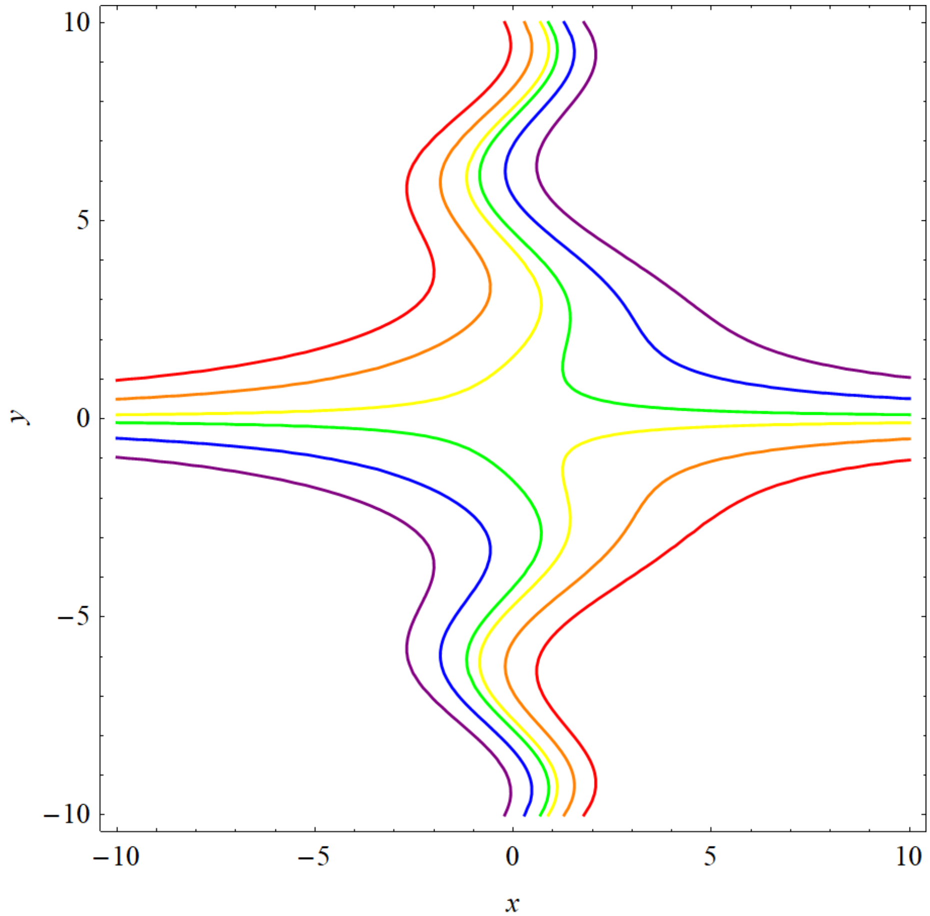
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_1$$

Therefore,

$$xy + y \cos y - \sin y = C_1.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C_1 = -10$, $C_1 = -5$, $C_1 = -1$, $C_1 = 1$, $C_1 = 5$, and $C_1 = 10$, respectively.