Problem 29

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

$$e^x + (e^x \cot y + 2y \csc y)y' = 0$$

Solution

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(e^x) = 0 \neq \frac{\partial}{\partial x}(e^x \cot y + 2y \csc y) = e^x \cot y.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$e^x \mu + \mu(e^x \cot y + 2y \csc y)y' = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}(e^x \mu) = \frac{\partial}{\partial x}[\mu(e^x \cot y + 2y \csc y)].$$

Expand both sides.

$$e^x \frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}(e^x \cot y + 2y \csc y) + \mu(e^x \cot y)$$

Assume that μ is only dependent on y: $\mu = \mu(y)$.

$$e^x \frac{d\mu}{dy} = \mu(e^x \cot y)$$

$$\frac{d\mu}{dy} = \mu(\cot y)$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = \cot y \, dy$$

Integrate both sides.

$$\ln \mu = \ln \sin y + C$$

Exponentiate both sides.

$$\mu = (\sin y)e^C$$

Taking e^C to be 1, an integrating factor is

$$\mu = \sin y.$$

Multiply both sides of the original ODE by $\sin y$.

$$e^x \sin y + (e^x \cos y + 2y)y' = 0$$

Because it's exact, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = e^x \sin y \tag{1}$$

$$\frac{\partial \psi}{\partial y} = e^x \cos y + 2y. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = e^x \sin y + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = e^x \cos y + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 2y \quad \rightarrow \quad f(y) = y^2.$$

As a result, a potential function is

$$\psi(x, y) = e^x \sin y + y^2.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

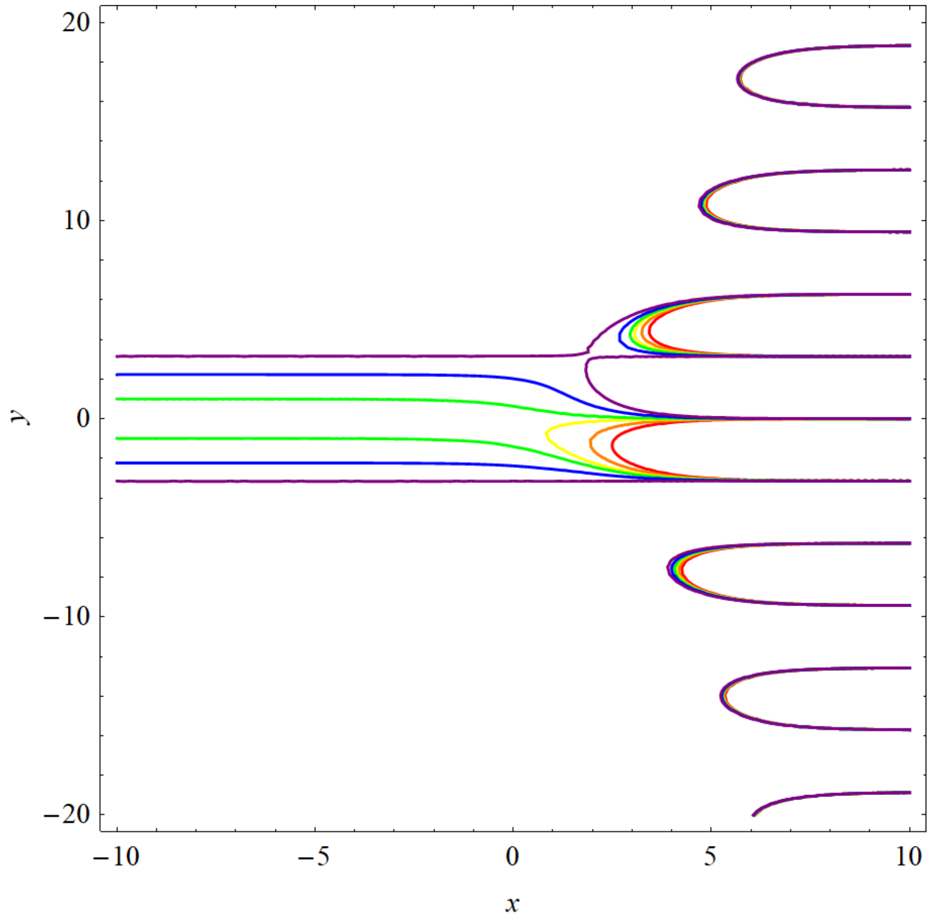
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_1$$

Therefore,

$$e^x \sin y + y^2 = C_1.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C_1 = -10$, $C_1 = -5$, $C_1 = -1$, $C_1 = 1$, $C_1 = 5$, and $C_1 = 10$, respectively.