Problem 31

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

$$\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)\frac{dy}{dx} = 0$$

Hint: See Problem 24.

Solution

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}\left(3x + \frac{6}{y}\right) = -\frac{6}{y^2} \neq \frac{\partial}{\partial x}\left(\frac{x^2}{y} + 3\frac{y}{x}\right) = 2\frac{x}{y} - 3\frac{y}{x^2}.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$\mu\left(3x + \frac{6}{y}\right) + \mu\left(\frac{x^2}{y} + 3\frac{y}{x}\right)\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}\left[\mu\left(3x + \frac{6}{y}\right)\right] = \frac{\partial}{\partial x}\left[\mu\left(\frac{x^2}{y} + 3\frac{y}{x}\right)\right].$$

Expand both sides.

$$\frac{\partial \mu}{\partial y}\left(3x + \frac{6}{y}\right) + \mu\left(-\frac{6}{y^2}\right) = \frac{\partial \mu}{\partial x}\left(\frac{x^2}{y} + 3\frac{y}{x}\right) + \mu\left(2\frac{x}{y} - 3\frac{y}{x^2}\right)$$

Following the hint, assume that μ is dependent on xy: $\mu = \mu(xy)$.

$$x\mu'(xy)\left(3x + \frac{6}{y}\right) + \mu\left(-\frac{6}{y^2}\right) = y\mu'(xy)\left(\frac{x^2}{y} + 3\frac{y}{x}\right) + \mu\left(2\frac{x}{y} - 3\frac{y}{x^2}\right)$$

$$\mu'(xy)\left(3x^2 + \frac{6x}{y}\right) = \mu'(xy)\left(x^2 + \frac{3y^2}{x}\right) + \mu\left(2\frac{x}{y} - 3\frac{y}{x^2} + \frac{6}{y^2}\right)$$

$$\mu'(xy)\left(2x^2 + \frac{6x}{y} - \frac{3y^2}{x}\right) = \mu\left(\frac{2x}{y} - \frac{3y}{x^2} + \frac{6}{y^2}\right)$$

$$\mu'(xy)xy\left(\frac{2x}{y} + \frac{6}{y^2} - \frac{3y}{x^2}\right) = \mu\left(\frac{2x}{y} - \frac{3y}{x^2} + \frac{6}{y^2}\right)$$

$$\mu'(xy)xy = \mu$$

Let $z = xy$ and solve this ODE by separating variables.

$$\frac{d\mu}{dz}z = \mu$$

$$\frac{d\mu}{\mu} = \frac{dz}{z}$$

Integrate both sides.

$$\ln \mu = \ln z + C$$

Exponentiate both sides.

$$\mu = (z)e^C$$

Taking e^C to be 1, an integrating factor is

$$\mu = z = xy.$$

Multiply both sides of the original ODE by xy.

$$(3x^2y + 6x) + (x^3 + 3y^2)\frac{dy}{dx} = 0$$

Because it's exact, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 3x^2y + 6x \tag{1}$$

$$\frac{\partial \psi}{\partial y} = x^3 + 3y^2. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = x^3y + 3x^2 + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = x^3 + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 3y^2 \quad \rightarrow \quad f(y) = y^3.$$

As a result, a potential function is

$$\psi(x, y) = x^3y + 3x^2 + y^3.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

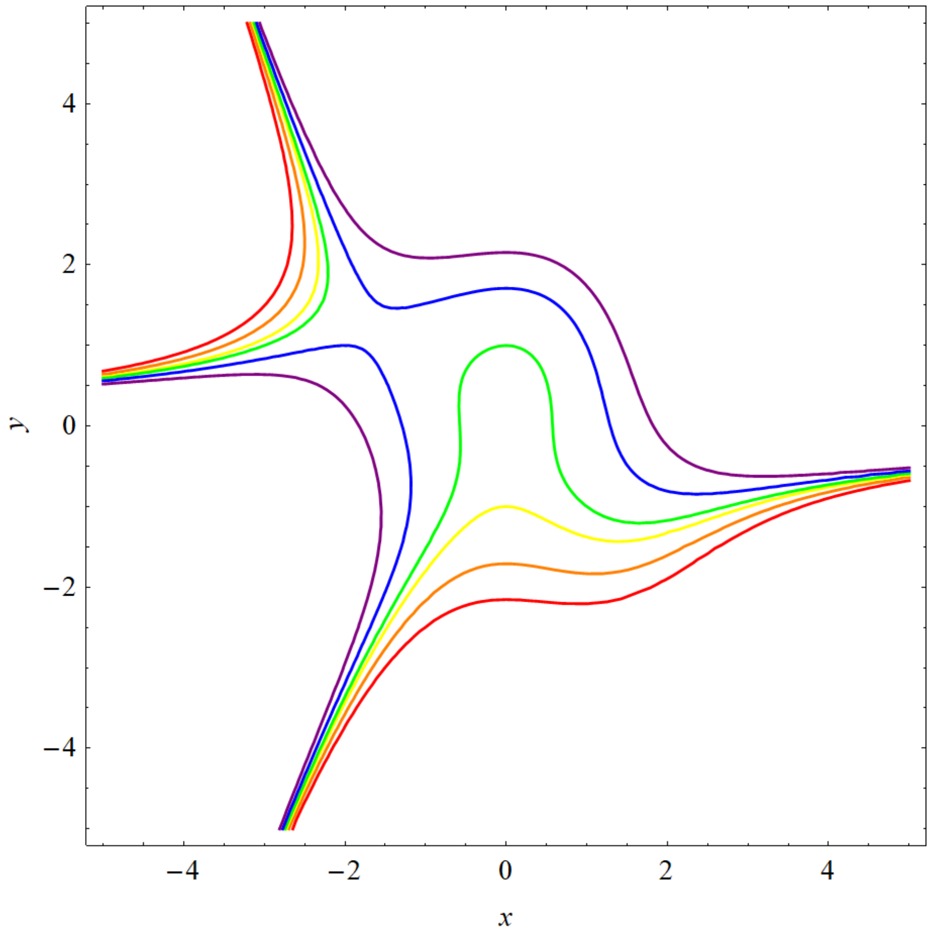
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_1$$

Therefore,

$$x^3y + 3x^2 + y^3 = C_1.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C_1 = -10$, $C_1 = -5$, $C_1 = -1$, $C_1 = 1$, $C_1 = 5$, and $C_1 = 10$, respectively.