Problem 32

Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

using the integrating factor $\mu(x, y) = [xy(2x + y)]^{-1}$. Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

Solution

Multiply both sides of the ODE by $\mu(x, y)$.

$$\left[\frac{3}{2x+y} + \frac{y}{x(2x+y)}\right] + \left[\frac{x}{y(2x+y)} + \frac{1}{2x+y}\right] y' = 0$$

This ODE is exact because

$$\frac{\partial}{\partial y}\left[\frac{3}{2x+y} + \frac{y}{x(2x+y)}\right] = \frac{\partial}{\partial x}\left[\frac{x}{y(2x+y)} + \frac{1}{2x+y}\right] = -\frac{1}{(2x+y)^2}.$$

That means there exists a potential function $\psi = \psi(x, y)$ which satisfies

$$\frac{\partial \psi}{\partial x} = \frac{3}{2x+y} + \frac{y}{x(2x+y)} \tag{1}$$

$$\frac{\partial \psi}{\partial y} = \frac{x}{y(2x+y)} + \frac{1}{2x+y}. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\int^x \left.\frac{\partial \psi}{\partial x}\right|_{x=s} ds = \int^x \frac{3}{2s+y}\,ds + \int^x \frac{y}{s(2s+y)}\,ds + f(y)$$

Here $f(y)$ is an arbitrary function of y.

$$\begin{aligned}\psi(x,y) &= \int^x \frac{3}{2s+y}\,ds + \int^x \left(\frac{1}{s} + \frac{-2}{2s+y}\right) ds + f(y) \\ &= \int^x \frac{ds}{2s+y} + \int^x \frac{ds}{s} + f(y)\end{aligned}$$

Use the substitution $u = 2s + y$ and $du = 2\,ds$ in the first integral.

$$\begin{aligned}\psi(x,y) &= \int^{2x+y} \frac{du/2}{u} + \int^x \frac{ds}{s} + f(y) \\ &= \frac{1}{2}\ln(2x+y) + \ln x + f(y)\end{aligned}$$

Differentiate both sides with respect to y.

$$\psi_y(x,y) = \frac{1}{2(2x+y)} + f'(y)$$

Comparing this to equation (2), we see that

$$\frac{1}{2(2x+y)} + f'(y) = \frac{x}{y(2x+y)} + \frac{1}{2x+y} \quad \rightarrow \quad f'(y) = \frac{x}{y(2x+y)} + \frac{1}{2(2x+y)} = \frac{2x+y}{2y(2x+y)} = \frac{1}{2y},$$

which means

$$f(y) = \frac{1}{2}\ln y.$$

As a result, a potential function is

$$\psi(x, y) = \frac{1}{2}\ln(2x + y) + \ln x + \frac{1}{2}\ln y.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_1$$

$$\frac{1}{2}\ln(2x + y) + \ln x + \frac{1}{2}\ln y = C_1$$

$$\ln(2x + y) + 2\ln x + \ln y = 2C_1$$

$$\ln(2x + y) + \ln x^2 + \ln y = 2C_1$$

$$\ln[(2x + y)(x^2)(y)] = 2C_1$$

$$x^2 y(2x + y) = e^{2C_1}$$

$$2x^3 y + x^2 y^2 = e^{2C_1}$$

$$x^3 y + \frac{1}{2}x^2 y^2 = \frac{1}{2}e^{2C_1}$$

Therefore, using a new constant c for the right side,

$$x^3 y + \frac{1}{2}x^2 y^2 = c,$$

which is the same answer obtained in Example 4 of the textbook.