Problem 4

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

$$(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$$

Solution

The ODE is exact because

$$\frac{\partial}{\partial y}(2xy^2 + 2y) = \frac{\partial}{\partial x}(2x^2y + 2x) = 4xy + 2.$$

That means there exists a potential function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = 2xy^2 + 2y \tag{1}$$

$$\frac{\partial \psi}{\partial y} = 2x^2y + 2x. \tag{2}$$

Integrate both sides of equation (1) partially with respect to x to get ψ.

$$\psi(x, y) = x^2y^2 + 2xy + f(y)$$

Here f is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = 2x^2y + 2x + f'(y)$$

Comparing this to equation (2), we see that

$$f'(y) = 0 \quad \to \quad f(y) = 0.$$

As a result, a potential function is

$$\psi(x, y) = x^2y^2 + 2xy.$$

Notice that by substituting equations (1) and (2), the ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

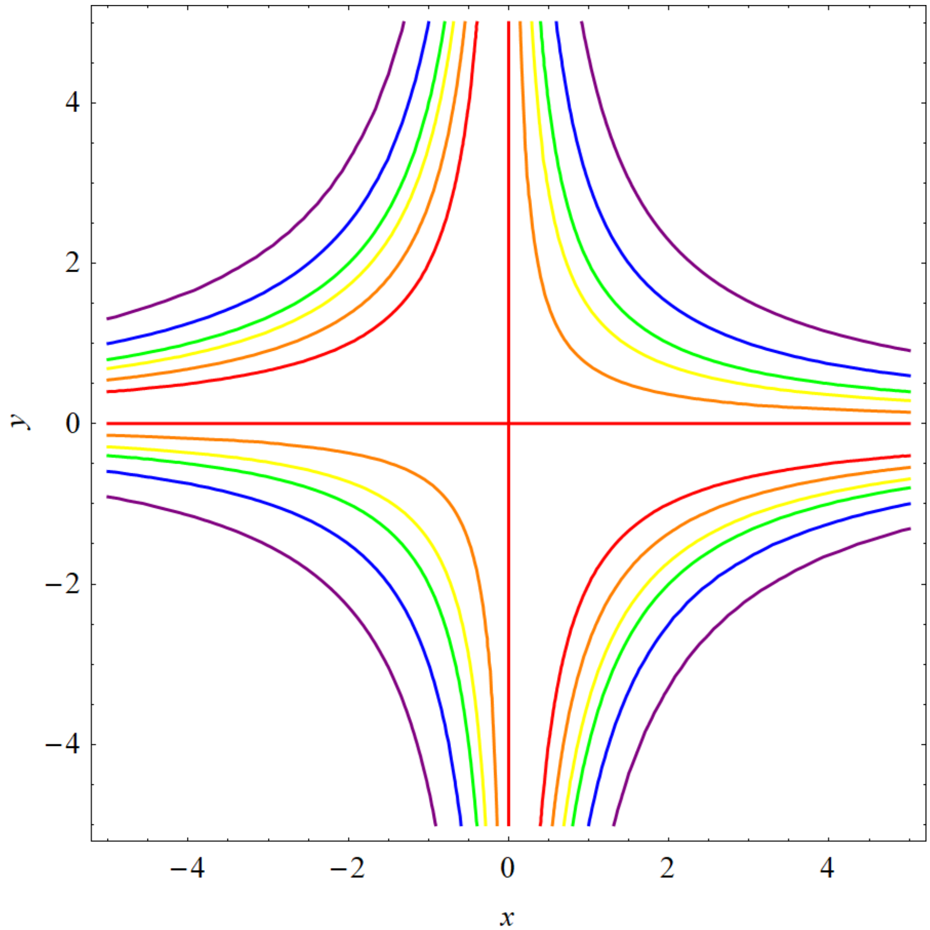
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$x^2y^2 + 2xy = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = 0$, $C = 2$, $C = 5$, $C = 8$, $C = 15$, and $C = 30$, respectively.