Problem 3

In each of Problems 3 through 6, let $\phi_0(t) = 0$ and define $\{\phi_n(t)\}$ by the method of successive approximations

(a) Determine $\phi_n(t)$ for an arbitrary value of n.

(b) Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.

(c) Express $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ in terms of elementary functions; that is, solve the given initial value problem.

(d) Plot $|\phi(t) - \phi_n(t)|$ for $n = 1, \ldots, 4$. For each of $\phi_1(t), \ldots, \phi_4(t)$, estimate the interval in which it is a reasonably good approximation to the actual solution.

$$y' = 2(y+1), \qquad y(0) = 0$$

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to t.

$$\begin{aligned} \frac{dy}{dt} &= 2(y+1) \\ \int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t 2[y(s)+1]\,ds \\ y(t) - y(0) &= \int_0^t 2[y(s)+1]\,ds \\ y(t) &= 2\int_0^t [y(s)+1]\,ds \end{aligned}$$

Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

$$y_{n+1}(t) = 2\int_0^t [y_n(s)+1]\,ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned} y_1(t) &= 2\int_0^t [y_0(s)+1]\,ds = 2\int_0^t ds = 2t \\ y_2(t) &= 2\int_0^t [y_1(s)+1]\,ds = 2\int_0^t (2s+1)\,ds = 2t + 2t^2 \\ y_3(t) &= 2\int_0^t [y_2(s)+1]\,ds = 2\int_0^t (2s+2s^2+1)\,ds = 2t + 2t^2 + \frac{4t^3}{3} \\ y_4(t) &= 2\int_0^t [y_3(s)+1]\,ds = 2\int_0^t \left(2s+2s^2+\frac{4s^3}{3}+1\right) ds = 2t + 2t^2 + \frac{4t^3}{3} + \frac{2t^4}{3} \\ y_5(t) &= 2\int_0^t [y_4(s)+1]\,ds = 2\int_0^t \left(2s+2s^2+\frac{4s^3}{3}+\frac{2s^4}{3}+1\right) ds = 2t + 2t^2 + \frac{4t^3}{3} + \frac{2t^4}{3} + \frac{4t^5}{15} \\ y_6(t) &= 2\int_0^t [y_5(s)+1]\,ds = 2\int_0^t \left(2s+2s^2+\frac{4s^3}{3}+\frac{2s^4}{3}+\frac{4s^5}{15}+1\right) ds = 2t + 2t^2 + \frac{4t^3}{3} + \frac{2t^4}{3} + \frac{4t^5}{15} + \frac{4t^6}{45} \\ &\vdots \end{aligned}$$

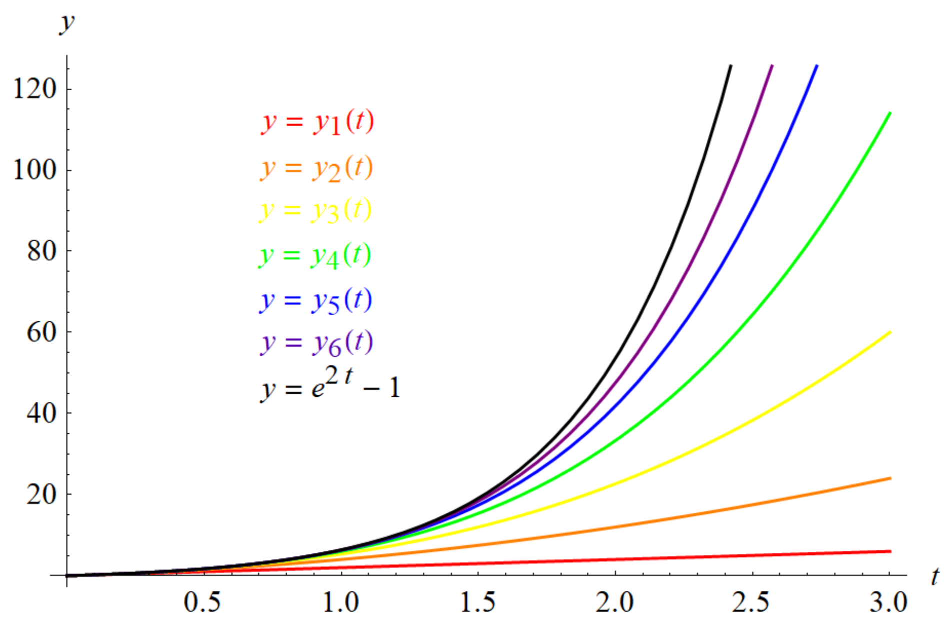
A formula for $y_n(t)$ can be deduced.

$$\begin{aligned}
y_n(t) &= 2t + 2t^2 + \frac{4t^3}{3} + \frac{2t^4}{3} + \frac{4t^5}{15} + \frac{4t^6}{45} + \cdots \\
&= 2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{2\cdot 3} + \frac{(2t)^4}{2\cdot 3\cdot 4} + \frac{(2t)^5}{2\cdot 3\cdot 4\cdot 5} + \frac{(2t)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots \\
&= \sum_{i=1}^{n} \frac{(2t)^i}{i!} \\
&= \sum_{i=0}^{n} \frac{(2t)^i}{i!} - 1
\end{aligned}$$

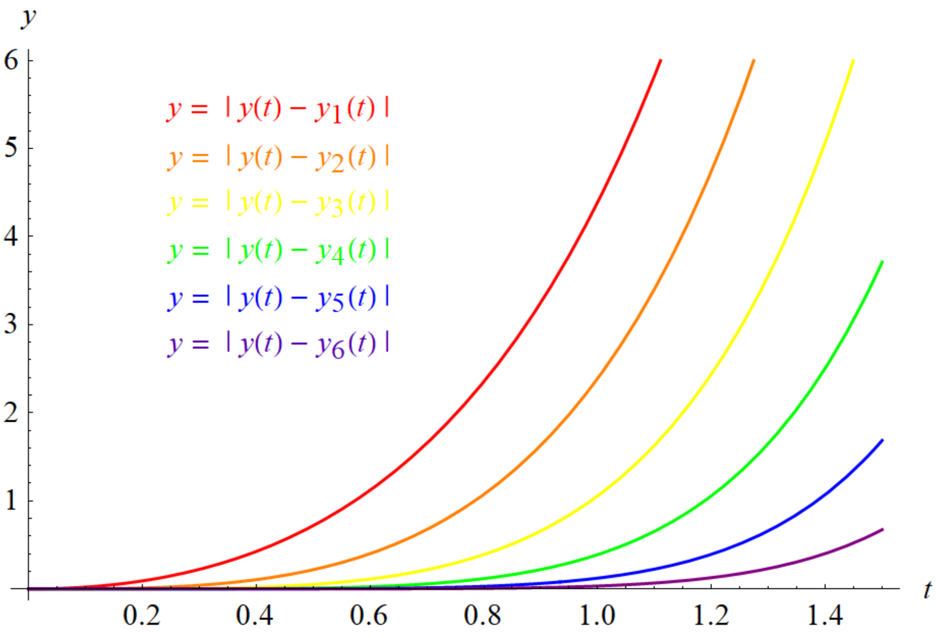
Obtain $y(t)$ now by taking the limit of $y_n(t)$ as $n \to \infty$.

$$\begin{aligned}
y(t) &= \lim_{n\to\infty} y_n(t) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(2t)^i}{i!} - 1 \\
&= \sum_{i=0}^{\infty} \frac{(2t)^i}{i!} - 1 \\
&= e^{2t} - 1
\end{aligned}$$

Not only is this the solution to the integral equation, but it also satisfies the initial value problem.



Based on the graphs, $y_n(t)$ seems to approach $y(t)$ as n increases.



$y_1(t)$, $y_2(t)$, $y_3(t)$, $y_4(t)$, $y_5(t)$, and $y_6(t)$ are good approximations to $y(t)$ only up to about $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$, and $t = 1.2$, respectively.