Problem 6

In each of Problems 3 through 6, let $\phi_0(t) = 0$ and define $\{\phi_n(t)\}$ by the method of successive approximations

(a) Determine $\phi_n(t)$ for an arbitrary value of n.

(b) Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.

(c) Express $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ in terms of elementary functions; that is, solve the given initial value problem.

(d) Plot $|\phi(t) - \phi_n(t)|$ for $n = 1, \ldots, 4$. For each of $\phi_1(t), \ldots, \phi_4(t)$, estimate the interval in which it is a reasonably good approximation to the actual solution.

$$y' = y + 1 - t, \qquad y(0) = 0$$

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to t.

$$\begin{aligned} \frac{dy}{dt} &= y + 1 - t \\ \int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t [y(s) + 1 - s]\, ds \\ y(t) - y(0) &= \int_0^t [y(s) + 1 - s]\, ds \\ y(t) &= \int_0^t [y(s) + 1 - s]\, ds \end{aligned}$$

Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

$$y_{n+1}(t) = \int_0^t [y_n(s) + 1 - s]\, ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned} y_1(t) &= \int_0^t [y_0(s) + 1 - s]\, ds = \int_0^t (1 - s)\, ds = t - \frac{t^2}{2} \\ y_2(t) &= \int_0^t [y_1(s) + 1 - s]\, ds = \int_0^t \left(s - \frac{s^2}{2} + 1 - s\right) ds = t - \frac{t^3}{6} \\ y_3(t) &= \int_0^t [y_2(s) + 1 - s]\, ds = \int_0^t \left(s - \frac{s^3}{6} + 1 - s\right) ds = t - \frac{t^4}{24} \\ y_4(t) &= \int_0^t [y_3(s) + 1 - s]\, ds = \int_0^t \left(s - \frac{s^4}{24} + 1 - s\right) ds = t - \frac{t^5}{120} \\ y_5(t) &= \int_0^t [y_4(s) + 1 - s]\, ds = \int_0^t \left(s - \frac{s^5}{120} + 1 - s\right) ds = t - \frac{t^6}{720} \\ y_6(t) &= \int_0^t [y_5(s) + 1 - s]\, ds = \int_0^t \left(s - \frac{s^6}{720} + 1 - s\right) ds = t - \frac{t^7}{5040} \\ &\vdots \end{aligned}$$

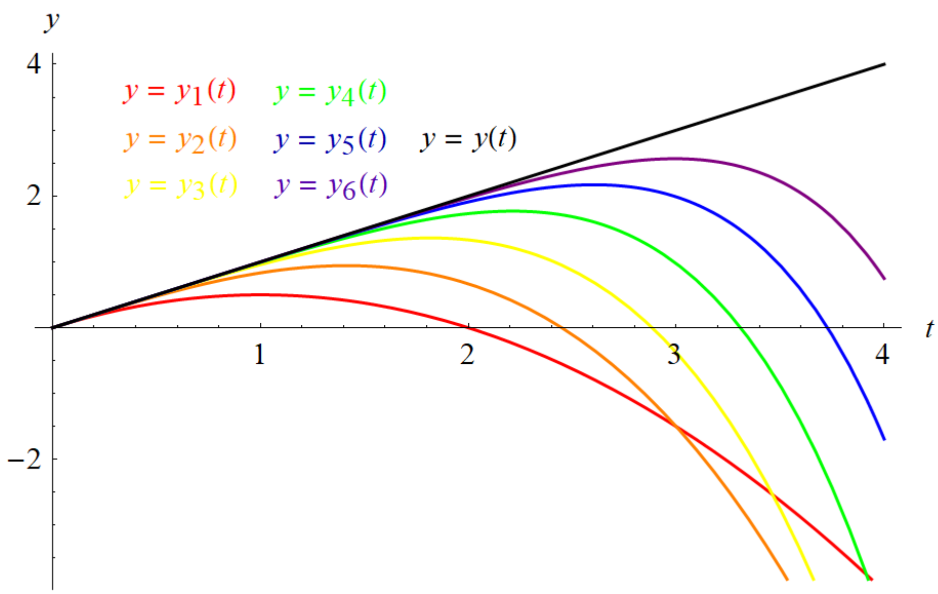
A formula for $y_n(t)$ can be deduced.

$$y_n(t) = t - \frac{t^{n+1}}{(n+1)!}$$

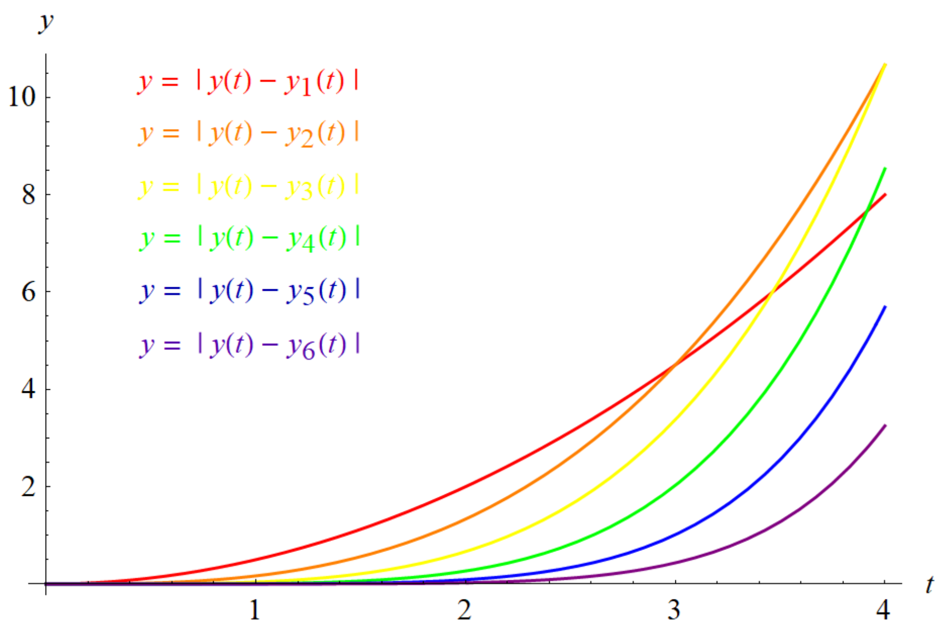
Obtain $y(t)$ now by taking the limit of $y_n(t)$ as $n \to \infty$.

$$\begin{aligned} y(t) &= \lim_{n\to\infty} y_n(t) \\ &= t - \underbrace{\lim_{n\to\infty} \frac{t^{n+1}}{(n+1)!}}_{=0} \\ &= t \end{aligned}$$

Not only is this the solution to the integral equation, but it also satisfies the initial value problem.



Based on the graphs, $y_n(t)$ seems to approach $y(t)$ as n increases.



$y_1(t)$, $y_2(t)$, $y_3(t)$, $y_4(t)$, $y_5(t)$, and $y_6(t)$ are good approximations to $y(t)$ only up to about $t = 1$, $t = 1.5$, $t = 2$, $t = 2.5$, $t = 3$, and $t = 3.5$, respectively.