# Problem 7

In each of Problems 7 and 8, let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem.

(a) Determine $\phi_n(t)$ for an arbitrary value of $n$.

(b) Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.

(c) Show that the sequence $\{\phi_n(t)\}$ converges.

$$y' = ty + 1, \qquad y(0) = 0$$

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### Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to $t$.

$$\begin{aligned}
\frac{dy}{dt} &= ty + 1 \\
\int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t [sy(s) + 1]\, ds \\
y(t) - y(0) &= \int_0^t [sy(s) + 1]\, ds \\
y(t) &= \int_0^t [sy(s) + 1]\, ds
\end{aligned}$$

Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

$$y_{n+1}(t) = \int_0^t [sy_n(s) + 1]\, ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned}
y_1(t) &= \int_0^t [sy_0(s) + 1]\, ds = \int_0^t ds = t \\
y_2(t) &= \int_0^t [sy_1(s) + 1]\, ds = \int_0^t (s^2 + 1)\, ds = t + \frac{t^3}{3} \\
y_3(t) &= \int_0^t [sy_2(s) + 1]\, ds = \int_0^t \left(s^2 + \frac{s^4}{3} + 1\right) ds = t + \frac{t^3}{3} + \frac{t^5}{15} \\
y_4(t) &= \int_0^t [sy_3(s) + 1]\, ds = \int_0^t \left(s^2 + \frac{s^4}{3} + \frac{s^6}{15} + 1\right) ds = t + \frac{t^3}{3} + \frac{t^5}{15} + \frac{t^7}{105} \\
y_5(t) &= \int_0^t [sy_4(s) + 1]\, ds = \int_0^t \left(s^2 + \frac{s^4}{3} + \frac{s^6}{15} + \frac{s^8}{105} + 1\right) ds = t + \frac{t^3}{3} + \frac{t^5}{15} + \frac{t^7}{105} + \frac{t^9}{945} \\
y_6(t) &= \int_0^t [sy_5(s) + 1]\, ds = \int_0^t \left(s^2 + \frac{s^4}{3} + \frac{s^6}{15} + \frac{s^8}{105} + \frac{s^{10}}{945} + 1\right) ds = t + \frac{t^3}{3} + \frac{t^5}{15} + \frac{t^7}{105} + \frac{t^9}{945} + \frac{t^{11}}{10\,395} \\
&\ \ \vdots
\end{aligned}$$

A formula for $y_n(t)$ can be deduced.

$$\begin{aligned} y_n(t) &= t + \frac{t^3}{3} + \frac{t^5}{15} + \frac{t^7}{105} + \frac{t^9}{945} + \frac{t^{11}}{10\,395} + \cdots \\ &= \frac{t}{1} + \frac{t^3}{1\cdot 3} + \frac{t^5}{1\cdot 3\cdot 5} + \frac{t^7}{1\cdot 3\cdot 5\cdot 7} + \frac{t^9}{1\cdot 3\cdot 5\cdot 7\cdot 9} + \frac{t^{11}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11} \\ &= \sum_{i=1}^{n} \frac{t^{2i-1}}{(2i-1)!!} \\ &= \sum_{i=1}^{n} \frac{t^{2i-1}}{\frac{(2i)!}{2^i i!}} \\ &= \sum_{i=1}^{n} \frac{2^i i!}{(2i)!} t^{2i-1} \end{aligned}$$

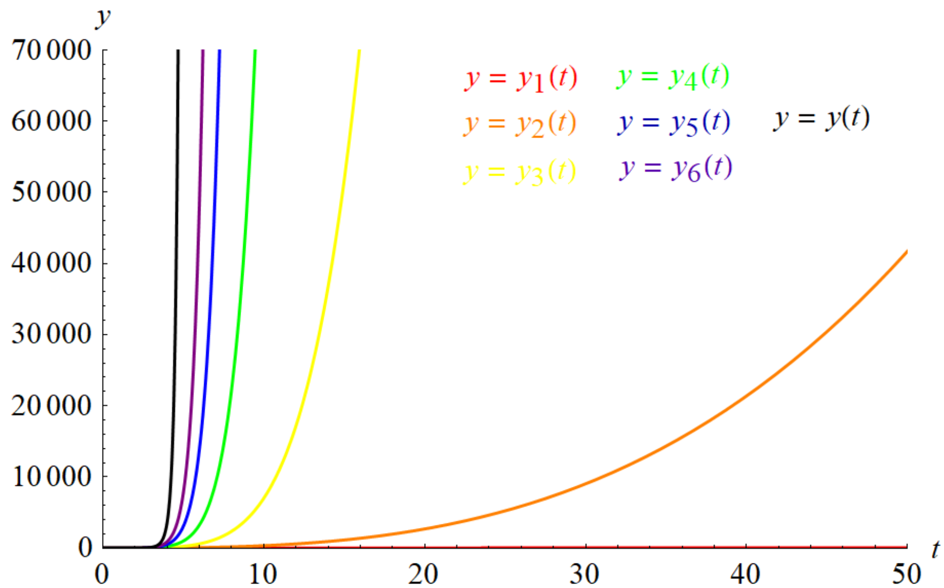
Obtain $y(t)$ now by taking the limit of $y_n(t)$ as $n \to \infty$.

$$\begin{aligned} y(t) &= \lim_{n\to\infty} y_n(t) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2^i i!}{(2i)!} t^{2i-1} \\ &= \sum_{i=1}^{\infty} \frac{2^i i!}{(2i)!} t^{2i-1} \end{aligned}$$

We can show that $y(t)$ converges by using the ratio test.

$$\begin{aligned} \lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| = \lim_{i\to\infty} \left| \frac{\frac{2^{i+1}(i+1)!}{[2(i+1)]!} t^{2(i+1)-1}}{\frac{2^i i!}{(2i)!} t^{2i-1}} \right| = \lim_{i\to\infty} \left| \frac{2^{i+1}}{2^i} \frac{(i+1)!}{i!} \frac{(2i)!}{(2i+2)!} \frac{t^{2i+1}}{t^{2i-1}} \right| &= \lim_{i\to\infty} \left| 2(i+1) \frac{1}{(2i+2)(2i+1)} t^2 \right| \\ &= 2t^2 \lim_{i\to\infty} \frac{i+1}{4i^2+6i+2} \\ &= t^2 \lim_{i\to\infty} \frac{1+\frac{1}{i}}{2i+\frac{3}{i}+\frac{1}{i^2}} \\ &= t^2 \lim_{i\to\infty} \frac{1}{2i} \\ &= 0 \end{aligned}$$

Since the limit of this ratio is less than 1, the series solution for $y(t)$ converges (assuming finite $t$).




Based on the graphs, $y_n(t)$ seems to approach $y(t)$ as $n$ increases.