Problem 12

In each of Problems 11 and 12, let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

(a) Calculate $\phi_1(t), \ldots, \phi_4(t)$, or (if necessary) Taylor approximations to these iterates. Keep terms up to order six.

(b) Plot the functions you found in part (a) and observe whether they appear to be converging.

$$y' = (3t^2 + 4t + 2)/2(y - 1), \qquad y(0) = 0$$

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to t.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{3t^2 + 4t + 2}{2(y - 1)} \\
\int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t \frac{3s^2 + 4s + 2}{2[y(s) - 1]}\, ds \\
y(t) - y(0) &= \int_0^t \frac{3s^2 + 4s + 2}{2[y(s) - 1]}\, ds \\
y(t) &= \int_0^t \frac{3s^2 + 4s + 2}{2[y(s) - 1]}\, ds
\end{aligned}$$

Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

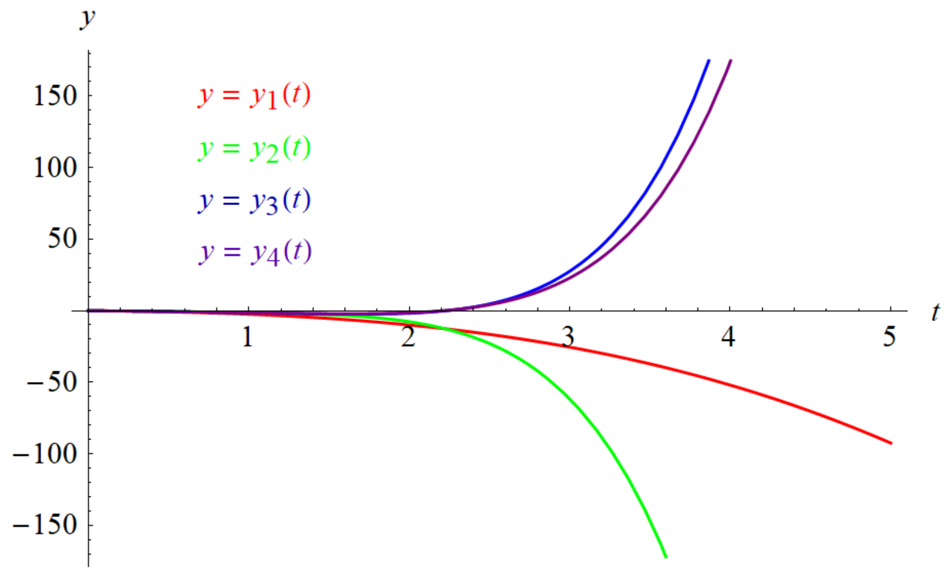
$$y_{n+1}(t) = \int_0^t \frac{3s^2 + 4s + 2}{2[y_n(s) - 1]}\, ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned}
y_1(t) &= \int_0^t \frac{3s^2 + 4s + 2}{2[y_0(s) - 1]}\, ds = \int_0^t \frac{3s^2 + 4s + 2}{2(-1)}\, ds = -t - t^2 - \frac{t^3}{2} \\
y_2(t) &= \int_0^t \frac{3s^2 + 4s + 2}{2[y_1(s) - 1]}\, ds = \int_0^t \frac{3s^2 + 4s + 2}{2\left(-s - s^2 - \frac{s^3}{2} - 1\right)}\, ds = -\int_0^t \frac{3s^2 + 4s + 2}{s^3 + 2s^2 + 2s + 2}\, ds \\
&= -\int_0^t \left(1 + s - \frac{s^2}{2} - s^3 + s^4 + \frac{s^5}{4} + \cdots\right) ds = -t - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{24} + \cdots \\
y_3(t) &= \int_0^t \frac{3s^2 + 4s + 2}{2[y_2(s) - 1]}\, ds = \int_0^t \frac{3s^2 + 4s + 2}{2\left(-s - \frac{s^2}{2} + \frac{s^3}{6} + \frac{s^4}{4} - \frac{s^5}{5} - \frac{s^6}{24} + \cdots - 1\right)}\, ds \\
&= \int_0^t \frac{3s^2 + 4s + 2}{-2 - 2s - s^2 + \frac{s^3}{3} + \frac{s^4}{2} - \frac{2s^5}{5} - \frac{s^6}{12} + \cdots}\, ds \\
&= \int_0^t \left(-1 - s + \frac{s^3}{3} - \frac{3s^4}{4} + \frac{8s^5}{15} + \cdots\right) ds \\
&= -t - \frac{t^2}{2} + \frac{t^4}{12} - \frac{3t^5}{20} + \frac{4t^6}{45} + \cdots
\end{aligned}$$

$$
\begin{aligned}
y_4(t) &= \int_0^t \frac{3s^2+4s+2}{2[y_3(s)-1]}\,ds = \int_0^t \frac{3s^2+4s+2}{2\left(-s-\frac{s^2}{2}+\frac{s^4}{12}-\frac{3s^5}{20}+\frac{4s^6}{45}+\cdots-1\right)}\,ds \\
&= \int_0^t \frac{3s^2+4s+2}{-2-2s-s^2+\frac{s^4}{6}-\frac{3s^5}{10}+\frac{8s^6}{45}+\cdots}\,ds \\
&= \int_0^t \left(-1-s+\frac{s^3}{2}-\frac{7s^4}{12}+\frac{2s^5}{5}+\cdots\right) ds \\
&= -t-\frac{t^2}{2}+\frac{t^4}{8}-\frac{7t^5}{60}+\frac{t^6}{15}+\cdots
\end{aligned}
$$

Below is a plot of these approximations (up to order 6) versus t.



They don't appear to converge.