Problem 14

Consider the sequence $\phi_n(x) = 2nxe^{-nx^2}$, $0 \le x \le 1$.

(a) Show that $\lim_{n\to\infty} \phi_n(x) = 0$ for $0 \le x \le 1$; hence

$$\int_0^1 \lim_{n\to\infty} \phi_n(x)\,dx = 0.$$

(b) Show that $\int_0^1 2nxe^{-nx^2}\,dx = 1 - e^{-n}$; hence

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)\,dx = 1.$$

Thus, in this example,

$$\lim_{n\to\infty} \int_a^b \phi_n(x)\,dx \neq \int_a^b \lim_{n\to\infty} \phi_n(x)\,dx,$$

even though $\lim_{n\to\infty} \phi_n(x)$ exists and is continuous.

Solution

Part (a)

$$\begin{aligned}
\lim_{n\to\infty} \phi_n(x) &= \lim_{n\to\infty} 2nxe^{-nx^2} \\
&= \lim_{n\to\infty} \frac{2nx}{e^{nx^2}} \\
&\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{n\to\infty} \frac{\frac{\partial}{\partial n}(2nx)}{\frac{\partial}{\partial n}(e^{nx^2})} \\
&= \lim_{n\to\infty} \frac{2x}{x^2 e^{nx^2}} \\
&= \frac{2}{x} \lim_{n\to\infty} \frac{1}{e^{nx^2}} \\
&= 0
\end{aligned}$$

Integrate both sides with respect to x from 0 to 1.

$$\int_0^1 \lim_{n\to\infty} \phi_n(x)\,dx = \int_0^1 0\,dx$$

Therefore,

$$\int_0^1 \lim_{n\to\infty} \phi_n(x)\,dx = 0.$$

Part (b)

$$\int_0^1 \phi_n(x)\,dx = \int_0^1 2nxe^{-nx^2}\,dx$$

Make the following substitution.

$$u = nx^2$$
$$du = 2nx\,dx$$

Then

$$\begin{aligned}\int_0^1 \phi_n(x)\,dx &= \int_0^n e^{-u}\,du \\ &= -e^{-u}\Big|_0^n \\ &= -e^{-n} + 1.\end{aligned}$$

Take the limit of both sides as $n \to \infty$.

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)\,dx = \lim_{n\to\infty} (-e^{-n} + 1)$$

Therefore,

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)\,dx = 1.$$