Problem 13

Let $\phi_n(x) = x^n$ for $0 \leq x \leq 1$ and show that

$$\lim_{n\to\infty} \phi_n(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

This example shows that a sequence of continuous functions may converge to a limit function that is discontinuous.

Solution

$$\begin{aligned} \lim_{n\to\infty} \phi_n(x) &= \lim_{n\to\infty} x^n \\ &= \lim_{n\to\infty} e^{\ln x^n} \\ &= \lim_{n\to\infty} e^{n \ln x} \end{aligned}$$

Suppose first that $0 \leq x < 1$. Then $\ln x$ is negative, and

$$\lim_{n\to\infty} \phi_n(x) = e^{-\infty} = 0.$$

Suppose secondly that $x = 1$. Then $\ln x = 0$, and

$$\lim_{n\to\infty} \phi_n(x) = e^0 = 1.$$

Therefore,

$$\lim_{n\to\infty} \phi_n(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$