Problem 14

Consider the sequence $\phi_n(x) = 2nxe^{-nx^2}$, $0 \le x \le 1$.

(a) Show that $\lim_{n\to\infty} \phi_n(x) = 0$ for $0 \le x \le 1$; hence

$$\int_0^1 \lim_{n\to\infty} \phi_n(x)\,dx = 0.$$

(b) Show that $\int_0^1 2nxe^{-nx^2}\,dx = 1 - e^{-n}$; hence

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)\,dx = 1.$$

Thus, in this example,

$$\lim_{n\to\infty} \int_a^b \phi_n(x)\,dx \neq \int_a^b \lim_{n\to\infty} \phi_n(x)\,dx,$$

even though $\lim_{n\to\infty} \phi_n(x)$ exists and is continuous.