Problem 5

In each of Problems 3 through 6, let $\phi_0(t) = 0$ and define $\{\phi_n(t)\}$ by the method of successive approximations

(a) Determine $\phi_n(t)$ for an arbitrary value of n.

(b) Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.

(c) Express $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ in terms of elementary functions; that is, solve the given initial value problem.

(d) Plot $|\phi(t) - \phi_n(t)|$ for $n = 1, \ldots, 4$. For each of $\phi_1(t), \ldots, \phi_4(t)$, estimate the interval in which it is a reasonably good approximation to the actual solution.

$$y' = -y/2 + t, \qquad y(0) = 0$$

Solution

Start by converting the initial value problem to an integral equation. Integrate both sides of the ODE from 0 to t.

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{y}{2} + t \\
\int_0^t \left.\frac{dy}{dt}\right|_{t=s} ds &= \int_0^t \left[-\frac{y(s)}{2} + s\right] ds \\
y(t) - y(0) &= \int_0^t \left[-\frac{y(s)}{2} + s\right] ds \\
y(t) &= \int_0^t \left[-\frac{y(s)}{2} + s\right] ds
\end{aligned}$$

Use the method of successive approximations to solve for $y(t)$. Consider the iteration scheme,

$$y_{n+1}(t) = \int_0^t \left[-\frac{y_n(s)}{2} + s\right] ds, \quad n \geq 0,$$

taking $y_0(t) = 0$ for the zeroth approximation. As a result,

$$\begin{aligned}
y_1(t) &= \int_0^t \left[-\frac{y_0(s)}{2} + s\right] ds = \int_0^t s\, ds = \frac{t^2}{2} \\
y_2(t) &= \int_0^t \left[-\frac{y_1(s)}{2} + s\right] ds = \int_0^t \left(-\frac{\frac{s^2}{2}}{2} + s\right) ds = \frac{t^2}{2} - \frac{t^3}{12} \\
y_3(t) &= \int_0^t \left[-\frac{y_2(s)}{2} + s\right] ds = \int_0^t \left(-\frac{\frac{s^2}{2} - \frac{s^3}{12}}{2} + s\right) ds = \frac{t^2}{2} - \frac{t^3}{12} + \frac{t^4}{96} \\
y_4(t) &= \int_0^t \left[-\frac{y_3(s)}{2} + s\right] ds = \int_0^t \left(-\frac{\frac{s^2}{2} - \frac{s^3}{12} + \frac{s^4}{96}}{2} + s\right) ds = \frac{t^2}{2} - \frac{t^3}{12} + \frac{t^4}{96} - \frac{t^5}{960}
\end{aligned}$$

$$y_5(t) = \int_0^t \left[-\frac{y_4(s)}{2} + s\right] ds = \int_0^t \left(-\frac{\frac{s^2}{2} - \frac{s^3}{12} + \frac{s^4}{96} - \frac{s^5}{960}}{2} + s\right) ds = \frac{t^2}{2} - \frac{t^3}{12} + \frac{t^4}{96} - \frac{t^5}{960} + \frac{t^6}{11\,520}$$

$$y_6(t) = \int_0^t \left[-\frac{y_5(s)}{2} + s\right] ds = \int_0^t \left(-\frac{\frac{s^2}{2} - \frac{s^3}{12} + \frac{s^4}{96} - \frac{s^5}{960} + \frac{s^6}{11\,520}}{2} + s\right) ds = \frac{t^2}{2} - \frac{t^3}{12} + \frac{t^4}{96} - \frac{t^5}{960} + \frac{t^6}{11\,520} - \frac{t^7}{161\,280}$$

$$\vdots$$

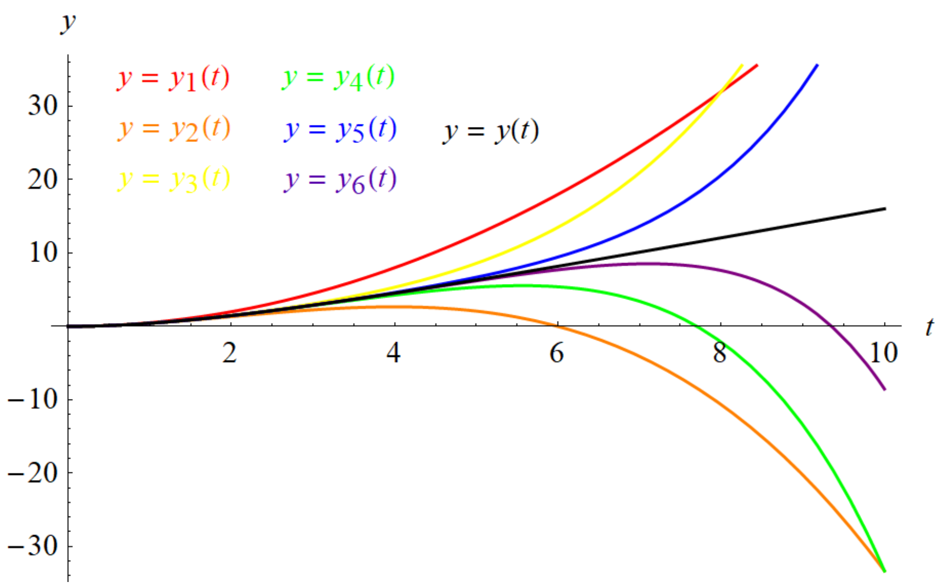
A formula for $y_n(t)$ can be deduced.

$$\begin{aligned}
y_n(t) &= \frac{t^2}{2} - \frac{t^3}{12} + \frac{t^4}{96} - \frac{t^5}{960} + \frac{t^6}{11\,520} - \frac{t^7}{161\,280} + \cdots \\
&= 2\frac{(-t)^2}{2\cdot 2!} + 2\frac{(-t)^3}{2^2\cdot 3!} + 2\frac{(-t)^4}{2^3\cdot 4!} + 2\frac{(-t)^5}{2^4\cdot 5!} + 2\frac{(-t)^6}{2^5\cdot 6!} + 2\frac{(-t)^7}{2^6\cdot 7!} + \cdots \\
&= 2^2\frac{(-t/2)^2}{2!} + 2^2\frac{(-t/2)^3}{3!} + 2^2\frac{(-t/2)^4}{4!} + 2^2\frac{(-t/2)^5}{5!} + 2^2\frac{(-t/2)^6}{6!} + 2^2\frac{(-t/2)^7}{7!} + \cdots \\
&= 4\left[\frac{(-t/2)^2}{2!} + \frac{(-t/2)^3}{3!} + \frac{(-t/2)^4}{4!} + \frac{(-t/2)^5}{5!} + \frac{(-t/2)^6}{6!} + \frac{(-t/2)^7}{7!} + \cdots\right] \\
&= 4\left[1 + \frac{(-t/2)}{1!} + \frac{(-t/2)^2}{2!} + \frac{(-t/2)^3}{3!} + \frac{(-t/2)^4}{4!} + \frac{(-t/2)^5}{5!} + \frac{(-t/2)^6}{6!} + \frac{(-t/2)^7}{7!} + \cdots\right] - 4 - 4\frac{(-t/2)}{1!} \\
&= 4\sum_{i=0}^{n} \frac{(-t/2)^i}{i!} - 4 + 2t
\end{aligned}$$

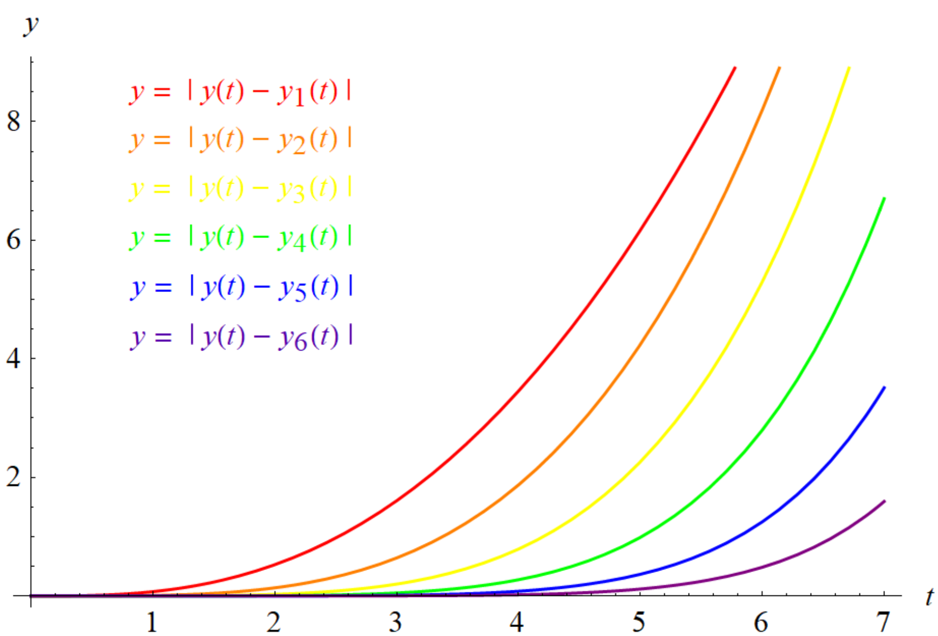
Obtain $y(t)$ now by taking the limit of $y_n(t)$ as $n \to \infty$.

$$\begin{aligned}
y(t) &= \lim_{n\to\infty} y_n(t) \\
&= \lim_{n\to\infty} 4\sum_{i=0}^{n} \frac{(-t/2)^i}{i!} - 4 + 2t \\
&= 4\sum_{i=0}^{\infty} \frac{(-t/2)^i}{i!} - 4 + 2t \\
&= 4e^{-t/2} - 4 + 2t
\end{aligned}$$

Not only is this the solution to the integral equation, but it also satisfies the initial value problem.



Based on the graphs, $y_n(t)$ seems to approach $y(t)$ as n increases.



$y_1(t)$, $y_2(t)$, $y_3(t)$, $y_4(t)$, $y_5(t)$, and $y_6(t)$ are good approximations to $y(t)$ only up to about $t = 1.5$, $t = 2.5$, $t = 3.5$, $t = 4$, $t = 5$, and $t = 6$, respectively.