Problem 3

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = \frac{2x+y}{3+3y^2-x}, \qquad y(0) = 0$$

Solution

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$(3+3y^2-x)\frac{dy}{dx} = 2x+y$$

$$(-2x-y) + (3+3y^2-x)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(-2x-y) = \frac{\partial}{\partial x}(3+3y^2-x) = -1.$$

That means there exists a potential function $\psi = \psi(x,y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = -2x-y \tag{2}$$

$$\frac{\partial \psi}{\partial y} = 3+3y^2-x. \tag{3}$$

Integrate both sides of equation (2) partially with respect to x to get ψ.

$$\psi(x,y) = -x^2 - xy + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x,y) = -x + f'(y)$$

Comparing this to equation (3), we see that

$$f'(y) = 3+3y^2 \quad \rightarrow \quad f(y) = 3y + y^3.$$

As a result, a potential function is

$$\psi(x,y) = -x^2 - xy + 3y + y^3.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x,y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

The general solution is then

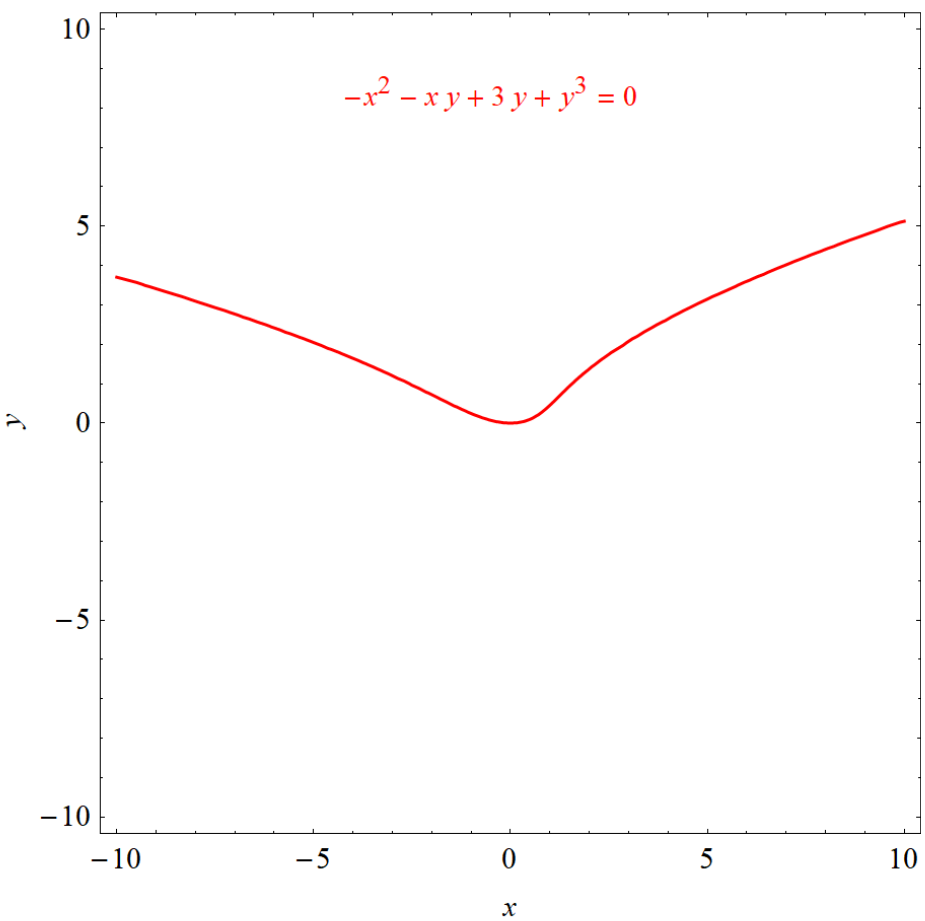
$$-x^2 - xy + 3y + y^3 = C.$$

Apply the initial condition $y(0) = 0$ now to determine C.

$$-0 - 0 + 0 + 0 = C \quad \rightarrow \quad C = 0$$

Therefore,

$$-x^2 - xy + 3y + y^3 = 0.$$



This figure illustrates the solution to the ODE in the xy-plane that passes through the point $(0, 0)$.