Problem 11

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$(x^2 + y) + (x + e^y)\frac{dy}{dx} = 0$$

Solution

This ODE is exact because
$$\frac{\partial}{\partial y}(x^2 + y) = \frac{\partial}{\partial x}(x + e^y) = 1.$$

That means there exists a potential function $\psi = \psi(x, y)$ which satisfies

$$\frac{\partial \psi}{\partial x} = x^2 + y \tag{1}$$

$$\frac{\partial \psi}{\partial y} = x + e^y. \tag{2}$$

Integrate both sides of equation (2) partially with respect to y to get ψ.

$$\psi(x, y) = xy + e^y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = y + f'(x)$$

Comparing this to equation (1), we see that

$$f'(x) = x^2 \quad \rightarrow \quad f(x) = \frac{x^3}{3}.$$

As a result, a potential function is

$$\psi(x, y) = xy + e^y + \frac{x^3}{3}.$$

Notice that by substituting equations (1) and (2), the original ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.
$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

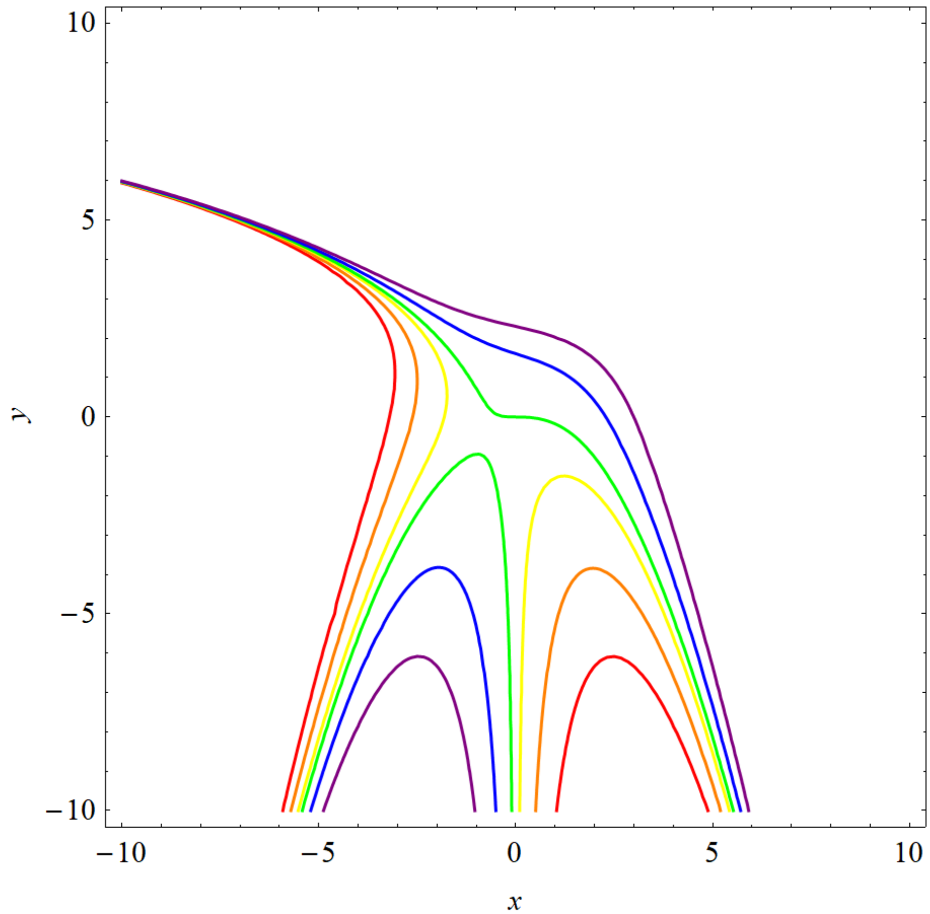
With it, equation (3) becomes
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$xy + e^y + \frac{x^3}{3} = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.