Problem 12

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} + y = \frac{1}{1+e^x}$$

Solution

Method Using an Integrating Factor I

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x ds\right) = e^x$$

Proceed with the multiplication.

$$e^x \frac{dy}{dx} + e^x y = \frac{e^x}{1+e^x}$$

The left side can be written as $d/dx(Iy)$ by the chain rule.

$$\frac{d}{dx}(e^x y) = \frac{e^x}{1+e^x}$$

Integrate both sides with respect to x.

$$e^x y = \int^x \frac{e^s}{1+e^s}\, ds + C$$

Make the substitution,

$$\begin{aligned} u &= 1 + e^s \\ du &= e^s\, ds. \end{aligned}$$

As a result,

$$\begin{aligned} e^x y &= \int^{1+e^x} \frac{du}{u} + C \\ &= \ln|u| \Big|^{1+e^x} + C \\ &= \ln(1+e^x) + C. \end{aligned}$$

Therefore,

$$y(x) = \frac{\ln(1+e^x) + C}{e^x}.$$

Method Using an Integrating Factor II

$$\frac{dy}{dx} + y = \frac{1}{1+e^x}$$

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$\left(y - \frac{1}{1+e^x}\right) + \frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}\left(y - \frac{1}{1+e^x}\right) = 1 \neq \frac{\partial}{\partial x}(1) = 0.$$

To solve it, we seek an integrating factor $\mu = \mu(x,y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$\left(y - \frac{1}{1+e^x}\right)\mu + \mu\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}\left[\left(y - \frac{1}{1+e^x}\right)\mu\right] = \frac{\partial}{\partial x}(\mu).$$

Expand both sides.

$$\mu + \left(y - \frac{1}{1+e^x}\right)\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$\mu = \frac{d\mu}{dx}$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = dx$$

Integrate both sides.

$$\ln \mu = x + C_1$$

Exponentiate both sides.

$$\begin{aligned}\mu &= e^{x+C_1} \\ &= e^x e^{C_1}\end{aligned}$$

Taking e^{C_1} to be 1, an integrating factor is

$$\mu = e^x.$$

Multiply both sides of equation (1) by e^x.

$$\left(e^x y - \frac{e^x}{1+e^x}\right) + e^x\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = e^x y - \frac{e^x}{1 + e^x} \tag{3}$$

$$\frac{\partial \psi}{\partial y} = e^x. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = e^x y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = e^x y + f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = -\frac{e^x}{1 + e^x} \quad \rightarrow \quad f(x) = -\ln(1 + e^x).$$

Consequently, a potential function is

$$\psi(x, y) = e^x y - \ln(1 + e^x).$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x} \, dx + \frac{\partial \psi}{\partial y} \, dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

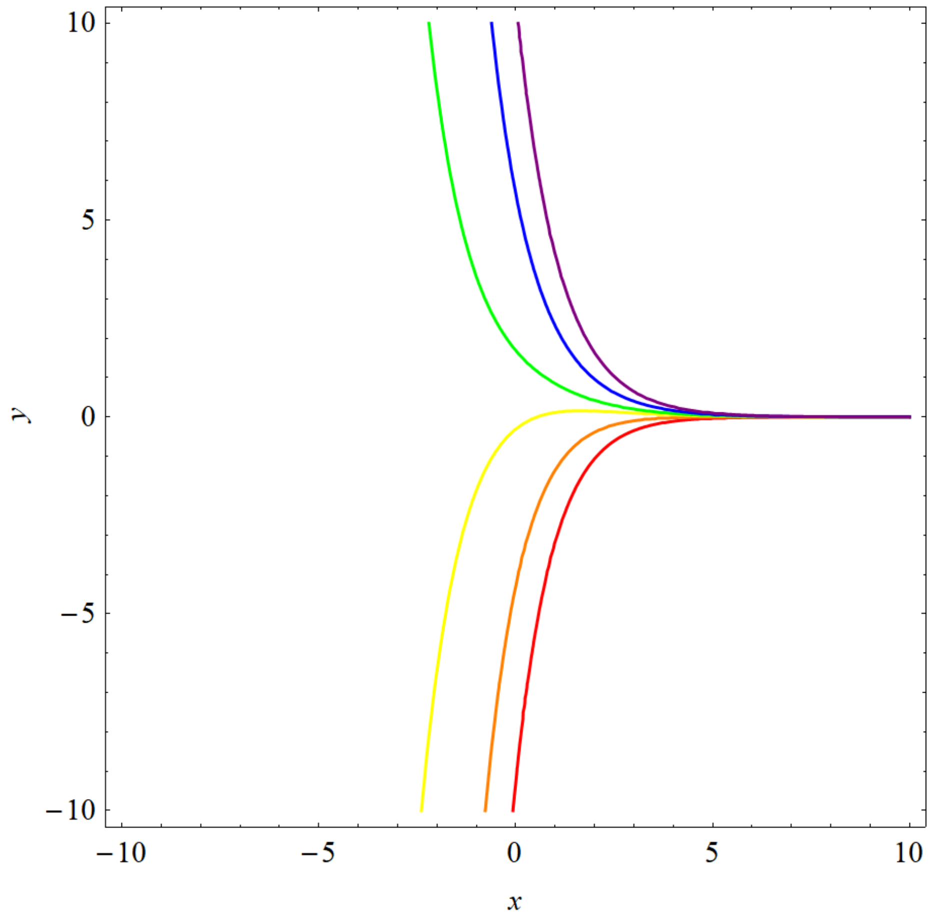
$$\psi(x, y) = C_2$$

Therefore,

$$e^x y - \ln(1 + e^x) = C_2,$$

or solving for y explicitly,

$$y(x) = \frac{\ln(1 + e^x) + C_2}{e^x}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.