Problem 14

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$(x+y)+(x+2y)\frac{dy}{dx}=0, \qquad y(2)=3$$

Solution

This ODE is exact because

$$\frac{\partial}{\partial y}(x+y)=\frac{\partial}{\partial x}(x+2y)=1.$$

That means there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x}=x+y \tag{1}$$

$$\frac{\partial \psi}{\partial y}=x+2y. \tag{2}$$

Integrate both sides of equation (2) partially with respect to y to get ψ.

$$\psi(x,y)=xy+y^2+f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x,y)=y+f'(x)$$

Comparing this to equation (1), we see that

$$f'(x)=x \quad \rightarrow \quad f(x)=\frac{x^2}{2}.$$

As a result, a potential function is

$$\psi(x,y)=xy+y^2+\frac{x^2}{2}.$$

Notice that by substituting equations (1) and (2), the original ODE can be written as

$$\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial y}\frac{dy}{dx}=0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi=\frac{\partial \psi}{\partial x}\,dx+\frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx}=\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

$$\frac{d\psi}{dx}=0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

The general solution is then

$$xy + y^2 + \frac{x^2}{2} = C.$$

Apply the boundary condition $y(2) = 3$ now to determine C.

$$(2)(3) + (3)^2 + \frac{(2)^2}{2} = C \quad \rightarrow \quad C = 17$$

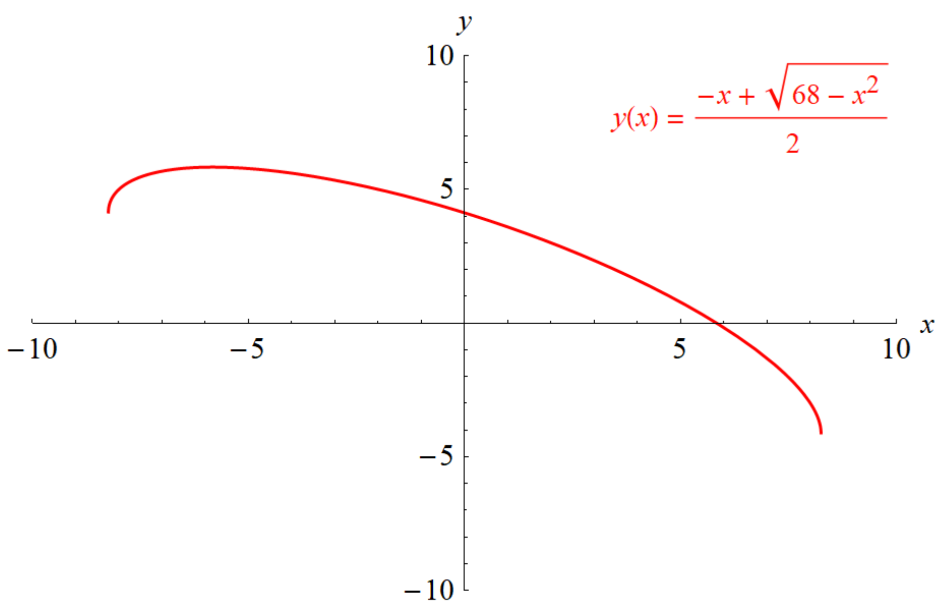
Therefore,

$$xy + y^2 + \frac{x^2}{2} = 17,$$

or solving for y explicitly,

$$y(x) = \frac{-x + \sqrt{68 - x^2}}{2}.$$

The plus sign was chosen so that the boundary condition remains satisfied.



This figure illustrates the solution to the ODE in the xy-plane that passes through the point $(2, 3)$.