Problem 15

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$(e^x + 1)\frac{dy}{dx} = y - ye^x$$

Solution

Method Using Separation of Variables

Factor the right side and then divide both sides by $1 + e^x$.

$$\frac{dy}{dx} = y\frac{1 - e^x}{1 + e^x}$$

Because the ODE is of the form $y' = f(x)g(y)$, it can be solved by separating variables.

$$\frac{dy}{y} = \frac{1 - e^x}{1 + e^x}\,dx$$

Integrate both sides.

$$\ln|y| = \int^x \frac{1 - e^s}{1 + e^s}\,ds + C$$

Make the following substitution.

$$u = e^s$$

$$du = e^s\,ds = u\,ds \quad \rightarrow \quad \frac{du}{u} = ds$$

As a result,

$$\begin{aligned}
\ln|y| &= \int^{e^x} \frac{1 - u}{1 + u}\frac{du}{u} + C \\
&= \int^{e^x} \frac{du}{(1 + u)u} - \int^{e^x} \frac{du}{1 + u} + C \\
&= \int^{e^x} \left(\frac{1}{u} - \frac{1}{1 + u}\right) du - \int^{e^x} \frac{du}{1 + u} + C \\
&= \ln|u|\Big|^{e^x} - \ln|1 + u|\Big|^{e^x} - \ln|1 + u|\Big|^{e^x} + C \\
&= \ln e^x - \ln(1 + e^x) - \ln(1 + e^x) + C \\
&= x - 2\ln(1 + e^x) + C
\end{aligned}$$

Bring the logarithms to the left side.

$$\ln|y| + \ln(1 + e^x)^2 = x + C$$

Combine them.

$$\ln|y|(1 + e^x)^2 = x + C$$

Exponentiate both sides.

$$|y|(1+e^x)^2 = e^{x+C}$$
$$|y| = \frac{e^C e^x}{(1+e^x)^2}$$

Introduce ± on the right side to remove the absolute value sign.

$$y(x) = \frac{\pm e^C e^x}{(1+e^x)^2}$$

Therefore, using a new constant A for $\pm e^C$,

$$\begin{aligned} y(x) &= \frac{Ae^x}{(1+e^x)^2} \\ &= \frac{A}{e^{-x}(1+e^x)^2} \\ &= \frac{A}{[e^{-x/2}(1+e^x)]^2} \\ &= \frac{A}{(e^{-x/2}+e^{x/2})^2} \\ &= \frac{A}{4\left(\frac{e^{x/2}+e^{-x/2}}{2}\right)^2} \\ &= \frac{A/4}{\cosh^2(x/2)} \\ &= \frac{B}{\cosh^2(x/2)}. \end{aligned}$$

Method Using an Integrating Factor

$$(e^x+1)\frac{dy}{dx}=y-ye^x$$

Bring all terms to the left side.

$$(e^x+1)\frac{dy}{dx}+(e^x-1)y=0$$

Divide both sides by e^x+1.

$$\frac{dy}{dx}+\frac{e^x-1}{e^x+1}y=0$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I=\exp\left(\int^x\frac{e^s-1}{e^s+1}\,ds\right)=e^{2\ln(1+e^x)-x}=e^{\ln(1+e^x)^2}e^{-x}=(1+e^x)^2e^{-x}$$

Proceed with the multiplication.

$$(e^x+1)^2e^{-x}\frac{dy}{dx}+(e^x-1)(e^x+1)e^{-x}y=0$$

$$(e^x+e^{-x}+2)\frac{dy}{dx}+(e^x-e^{-x})y=0$$

The left side can be written as $d/dx[(e^x+e^{-x}+2)y]$ by the chain rule.

$$\frac{d}{dx}[(e^x+e^{-x}+2)y]=0$$

Integrate both sides with respect to x.

$$(e^x+e^{-x}+2)y=C_1$$

Therefore,

$$\begin{aligned}
y(x)&=\frac{C_1}{e^x+e^{-x}+2}\\
&=\frac{C_1}{(e^{-x/2}+e^{x/2})^2}\\
&=\frac{C_1}{4\left(\frac{e^{x/2}+e^{-x/2}}{2}\right)^2}\\
&=\frac{C_1/4}{\cosh^2(x/2)}\\
&=\frac{C_2}{\cosh^2(x/2)}.
\end{aligned}$$

Method Using an Integrating Factor II

$$(e^x + 1)\frac{dy}{dx} = y - ye^x$$

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$(ye^x - y) + (e^x + 1)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(ye^x - y) = e^x - 1 \neq \frac{\partial}{\partial x}(e^x + 1) = e^x.$$

To solve it, we seek an integrating factor $\mu = \mu(x,y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(ye^x - y)\mu + (e^x + 1)\mu\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(ye^x - y)\mu] = \frac{\partial}{\partial x}[(e^x + 1)\mu].$$

Expand both sides.

$$(e^x - 1)\mu + (ye^x - y)\frac{\partial \mu}{\partial y} = e^x\mu + (e^x + 1)\frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$(e^x - 1)\mu = e^x\mu + (e^x + 1)\frac{d\mu}{dx}$$

$$-\mu = (e^x + 1)\frac{d\mu}{dx}$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = -\frac{dx}{e^x + 1}$$

Integrate both sides.

$$\begin{aligned}\ln \mu &= -\int^x \frac{ds}{e^s + 1} + C_3 \\ &= -\int^x \frac{e^{-s}}{1 + e^{-s}}\, ds + C_3\end{aligned}$$

Make the following substitution.

$$\begin{aligned} v &= 1 + e^{-s} \\ dv &= -e^{-s}\, ds\end{aligned}$$

As a result,

$$\begin{aligned}\ln \mu &= -\int^{1+e^{-x}} \frac{-dv}{v} + C_3 \\ &= \ln(1 + e^{-x}) + C_3\end{aligned}$$

Exponentiate both sides.

$$\mu = e^{\ln(1+e^{-x})+C_3}$$
$$= (1+e^{-x})e^{C_3}$$

Taking e^{C_3} to be 1, an integrating factor is

$$\mu = 1 + e^{-x}.$$

Multiply both sides of equation (1) by $1 + e^{-x}$.

$$(ye^x - y)(1+e^{-x}) + (e^x+1)(1+e^{-x})\frac{dy}{dx} = 0$$

$$(e^x y - e^{-x} y) + (e^x + e^{-x} + 2)\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = e^x y - e^{-x} y \tag{3}$$

$$\frac{\partial \psi}{\partial y} = e^x + e^{-x} + 2. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = e^x y + e^{-x} y + 2y + h(x)$$

Here $h(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = e^x y - e^{-x} y + h'(x)$$

Comparing this to equation (3), we see that

$$h'(x) = 0 \quad \rightarrow \quad h(x) = 0.$$

Consequently, a potential function is

$$\psi(x, y) = e^x y + e^{-x} y + 2y.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_4$$

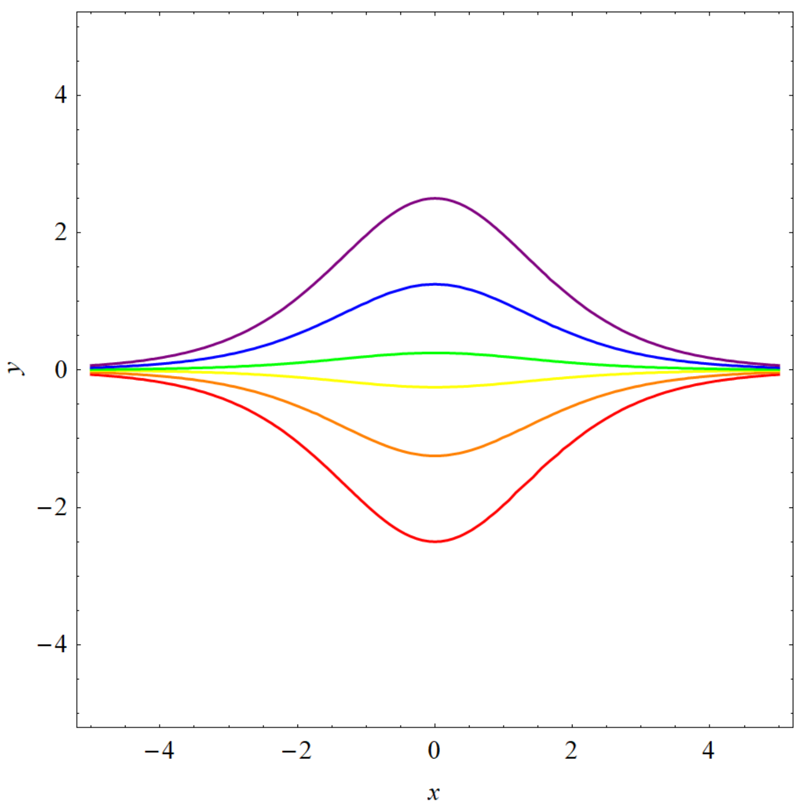
Therefore,

$$e^x y + e^{-x} y + 2y = C_4,$$

or solving for y explicitly,

$$(e^x + e^{-x} + 2)y = C_4$$

$$\begin{aligned} y(x) &= \frac{C_4}{e^x + e^{-x} + 2} \\ &= \frac{C_4}{(e^{-x/2} + e^{x/2})^2} \\ &= \frac{C_4}{4\left(\frac{e^{x/2}+e^{-x/2}}{2}\right)^2} \\ &= \frac{C_4/4}{\cosh^2(x/2)} \\ &= \frac{C_5}{\cosh^2(x/2)}. \end{aligned}$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.