Problem 19

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = \frac{3x^2 - 2y - y^3}{2x + 3xy^2}$$

Solution

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$(2x + 3xy^2)\frac{dy}{dx} = 3x^2 - 2y - y^3$$

$$(y^3 + 2y - 3x^2) + (2x + 3xy^2)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(y^3 + 2y - 3x^2) = \frac{\partial}{\partial x}(2x + 3xy^2) = 2 + 3y^2.$$

That means there exists a potential function $\psi = \psi(x, y)$ which satisfies

$$\frac{\partial \psi}{\partial x} = y^3 + 2y - 3x^2 \tag{2}$$

$$\frac{\partial \psi}{\partial y} = 2x + 3xy^2. \tag{3}$$

Integrate both sides of equation (3) partially with respect to y to get ψ.

$$\psi(x, y) = 2xy + xy^3 + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = 2y + y^3 + f'(x)$$

Comparing this to equation (2), we see that

$$f'(x) = -3x^2 \quad \rightarrow \quad f(x) = -x^3.$$

Consequently, a potential function is

$$\psi(x, y) = 2xy + xy^3 - x^3.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

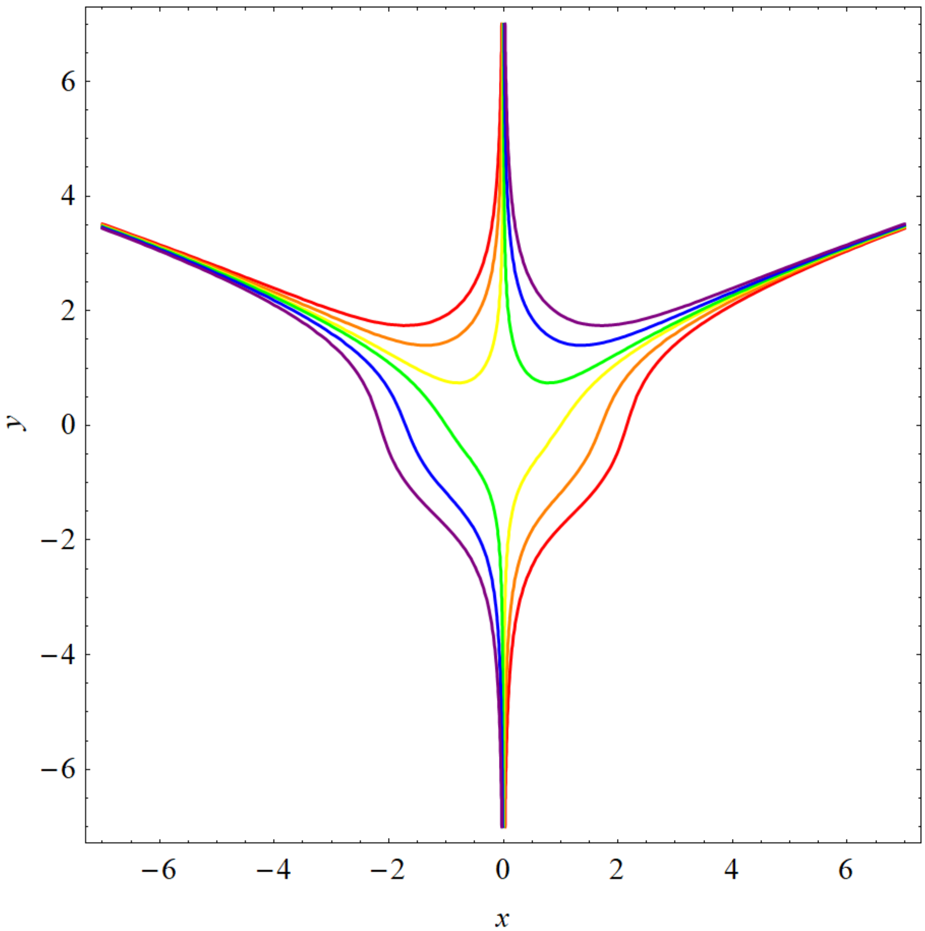
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$2xy + xy^3 - x^3 = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.