Problem 22

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = \frac{x^2 - 1}{y^2 + 1}, \qquad y(-1) = 1$$

Solution

Method Using Separation of Variables

Because the ODE is of the form $y' = f(x)/g(y)$, it can be solved by separating variables.

$$(y^2 + 1)\,dy = (x^2 - 1)\,dx$$

Integrate both sides.

$$\int (y^2 + 1)\,dy = \int (x^2 - 1)\,dx$$

$$\frac{y^3}{3} + y = \frac{x^3}{3} - x + C$$

The general solution is then

$$\frac{y^3}{3} + y - \frac{x^3}{3} + x = C.$$

Apply the boundary condition $y(-1) = 1$ now to determine C.

$$\frac{1}{3} + 1 - \frac{(-1)^3}{3} + (-1) = C \quad \to \quad C = \frac{2}{3}$$

Therefore,

$$\frac{y^3}{3} + y - \frac{x^3}{3} + x = \frac{2}{3}.$$

Method Using an Integrating Factor

$$\frac{dy}{dx} = \frac{x^2 - 1}{y^2 + 1}$$

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$(y^2 + 1)\frac{dy}{dx} = x^2 - 1$$

$$(1 - x^2) + (y^2 + 1)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(1 - x^2) = \frac{\partial}{\partial x}(y^2 + 1) = 0.$$

That means there exists a potential function $\psi = \psi(x, y)$ which satisfies

$$\frac{\partial \psi}{\partial x} = 1 - x^2 \tag{2}$$

$$\frac{\partial \psi}{\partial y} = y^2 + 1. \tag{3}$$

Integrate both sides of equation (2) partially with respect to x to get ψ.

$$\psi(x, y) = x - \frac{x^3}{3} + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = f'(y)$$

Comparing this to equation (3), we see that

$$f'(y) = y^2 + 1 \quad \rightarrow \quad f(y) = \frac{y^3}{3} + y.$$

As a result, a potential function is

$$\psi(x, y) = x - \frac{x^3}{3} + \frac{y^3}{3} + y.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_1$$

The general solution is then

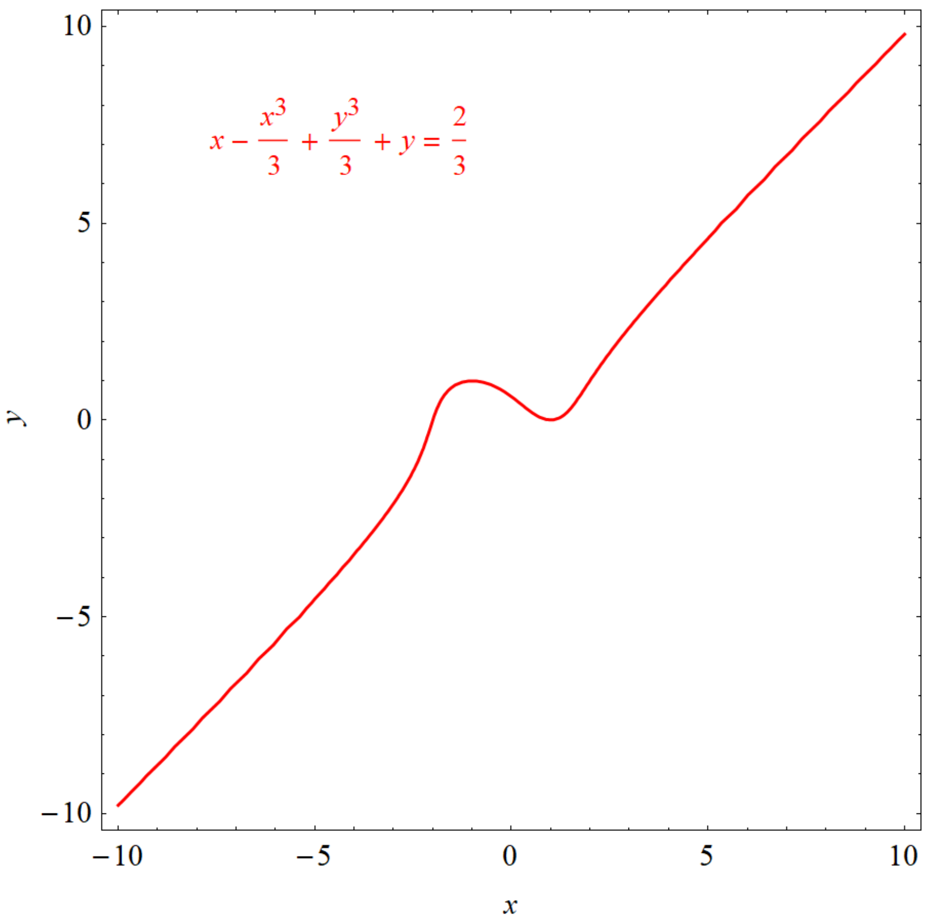
$$x - \frac{x^3}{3} + \frac{y^3}{3} + y = C_1.$$

Apply the boundary condition $y(-1) = 1$ now to determine C_1.

$$(-1) - \frac{(-1)^3}{3} + \frac{1}{3} + 1 = C_1 \quad \rightarrow \quad C_1 = \frac{2}{3}$$

Therefore,

$$x - \frac{x^3}{3} + \frac{y^3}{3} + y = \frac{2}{3}.$$



This figure illustrates the solution to the ODE in the xy-plane that passes through the point $(-1, 1)$.