Problem 27

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = \frac{x}{x^2y + y^3} \quad \textit{Hint:} \text{ Let } u = x^2.$$

Solution

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$(x^2y + y^3)\frac{dy}{dx} = x$$

$$-x + (x^2y + y^3)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(-x) = 0 \neq \frac{\partial}{\partial x}(x^2y + y^3) = 2xy.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$-x\mu + \mu(x^2y + y^3)\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}(-x\mu) = \frac{\partial}{\partial x}[\mu(x^2y + y^3)].$$

Expand both sides.

$$-x\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}(x^2y + y^3) + \mu(2xy)$$

Assume that μ is only dependent on y: $\mu = \mu(y)$.

$$-x\frac{d\mu}{dy} = \mu(2xy)$$

Divide both sides by x.

$$-\frac{d\mu}{dy} = \mu(2y)$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = -2y\,dy$$

Integrate both sides.

$$\ln \mu = -y^2 + C$$

Exponentiate both sides.

$$\begin{aligned} \mu &= e^{-y^2+C} \\ &= e^{-y^2}e^C \end{aligned}$$

Taking e^C to be 1, an integrating factor is

$$\mu = e^{-y^2}.$$

Multiply both sides of equation (1) by e^{-y^2}.

$$-xe^{-y^2} + (x^2ye^{-y^2} + y^3e^{-y^2})\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = -xe^{-y^2} \tag{3}$$

$$\frac{\partial \psi}{\partial y} = x^2ye^{-y^2} + y^3e^{-y^2}. \tag{4}$$

Integrate both sides of equation (3) partially with respect to x to get ψ.

$$\psi(x, y) = -\frac{x^2}{2}e^{-y^2} + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = x^2ye^{-y^2} + f'(y)$$

Comparing this to equation (4), we see that

$$\begin{aligned} f'(y) = y^3e^{-y^2} \quad \rightarrow \quad f(y) &= \int y^3e^{-y^2}\,dy \\ &= \int y^2 \cdot ye^{-y^2}\,dy \\ &= \int y^2 \cdot \frac{d}{dy}\left(-\frac{1}{2}e^{-y^2}\right)dy \\ &= y^2\left(-\frac{1}{2}e^{-y^2}\right) - \int (2y)\left(-\frac{1}{2}e^{-y^2}\right)dy \\ &= -\frac{y^2}{2}e^{-y^2} + \int ye^{-y^2}\,dy \\ &= -\frac{y^2}{2}e^{-y^2} - \frac{1}{2}e^{-y^2} \\ &= -\frac{e^{-y^2}}{2}(y^2+1) \end{aligned}$$

Consequently, a potential function is

$$\begin{aligned} \psi(x, y) &= -\frac{x^2}{2}e^{-y^2} - \frac{e^{-y^2}}{2}(y^2+1) \\ &= -\frac{e^{-y^2}}{2}(x^2+y^2+1). \end{aligned}$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x,y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

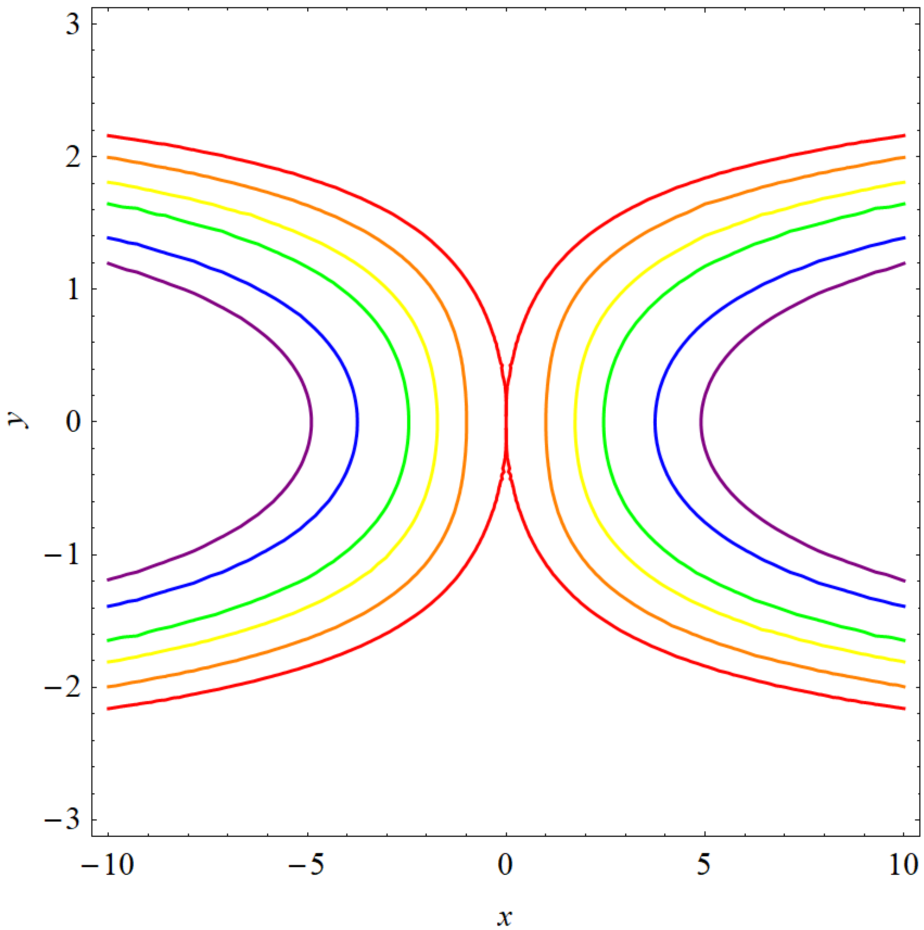
Integrate both sides with respect to x.

$$\psi(x,y) = C_1$$

$$-\frac{e^{-y^2}}{2}(x^2 + y^2 + 1) = C_1$$

Therefore, using a new constant $C_2 = -2C_1$ on the right side,

$$e^{-y^2}(x^2 + y^2 + 1) = C_2.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C_2 = 1$, $C_2 = 2$, $C_2 = 4$, $C_2 = 7$, $C_2 = 15$, and $C_2 = 25$, respectively.