Problem 30

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$(3y^2 + 2xy) - (2xy + x^2)\frac{dy}{dx} = 0$$

Solution

Solve for dy/dx.

$$\begin{aligned}\frac{dy}{dx} &= \frac{3y^2 + 2xy}{2xy + x^2} \\ &= \frac{3\frac{y^2}{x^2} + 2\frac{y}{x}}{2\frac{y}{x} + 1}\end{aligned}$$

Make the substitution $u = y/x$.

$$\frac{dy}{dx} = \frac{3u^2 + 2u}{2u + 1}$$

Differentiate both sides of the substitution with respect to x to find what dy/dx is in terms of this new variable.

$$\frac{du}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} \quad \to \quad x\frac{du}{dx} = \frac{dy}{dx} - \frac{y}{x} \quad \to \quad \frac{dy}{dx} = x\frac{du}{dx} + \frac{y}{x} = x\frac{du}{dx} + u$$

Consequently, the ODE that u satisfies is

$$x\frac{du}{dx} + u = \frac{3u^2 + 2u}{2u + 1}$$

$$\begin{aligned}x\frac{du}{dx} &= \frac{3u^2 + 2u}{2u + 1} - u \\ &= \frac{3u^2 + 2u - u(2u + 1)}{2u + 1} \\ &= \frac{u^2 + u}{2u + 1},\end{aligned}$$

which can be solved by separating variables.

$$\frac{2u + 1}{u^2 + u}\, du = \frac{dx}{x}$$

Integrate both sides.

$$\int \frac{2u + 1}{u^2 + u}\, du = \ln|x| + C$$

Let $v = u^2 + u$ so that $dv = (2u + 1)\, du$.

$$\int \frac{dv}{v} = \ln|x| + C$$

$$\ln|v| = \ln|x| + C$$

$$\ln|u^2 + u| = \ln|x| + C$$

Now that the integration is done, change back to y.

$$\ln\left|\frac{y^2}{x^2} + \frac{y}{x}\right| = \ln|x| + C$$

$$\ln\left|\frac{y^2}{x^2} + \frac{y}{x}\right| - \ln|x| = C$$

$$\ln\left|\frac{\frac{y^2}{x^2} + \frac{y}{x}}{x}\right| = C$$

$$\ln\left|\frac{y^2 + xy}{x^3}\right| = C$$

$$\left|\frac{y^2 + xy}{x^3}\right| = e^C$$

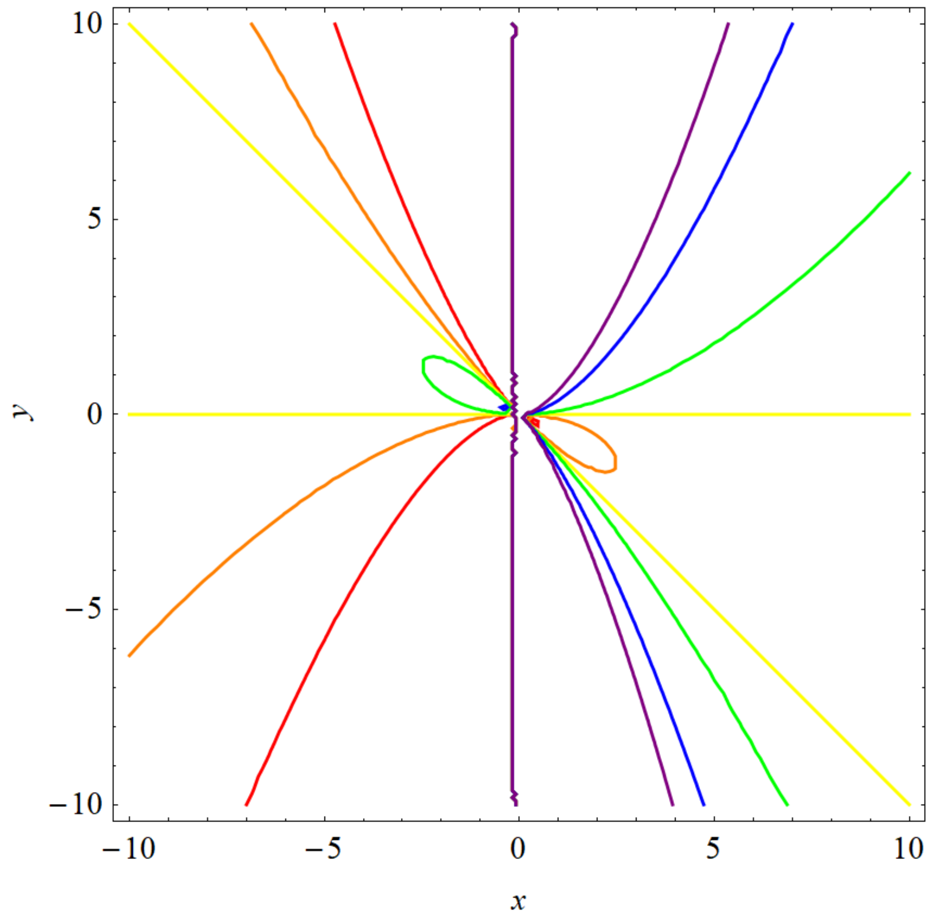
$$\frac{y^2 + xy}{x^3} = \pm e^C$$

Therefore, using a new constant A for $\pm e^C$,

$$\frac{y^2 + xy}{x^3} = A,$$

or solving for y explicitly,

$$y(x) = \frac{-x \pm \sqrt{x^2 + 4Ax^3}}{2}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $A = -0.5$, $A = -0.1$, $A = 0$, $A = 0.1$, $A = 0.5$, and $A = 1$, respectively.