Problem 32

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$xy' + y - y^2e^{2x} = 0, \qquad y(1) = 2$$

Solution

This is a Bernoulli equation. Bring the third term to the right side

$$xy' + y = y^2e^{2x}$$

and divide both sides by xy^2.

$$y^{-2}y' + \frac{1}{x}y^{-1} = \frac{e^{2x}}{x}$$

Make the substitution $u = y^{-1}$. Differentiate both sides of it with respect to x to write y' in terms of this new variable.

$$\frac{du}{dx} = -y^{-2} \cdot \frac{dy}{dx} \quad \rightarrow \quad y^{-2}y' = -\frac{du}{dx}$$

Consequently, the ODE that u satisfies is

$$-\frac{du}{dx} + \frac{1}{x}u = \frac{e^{2x}}{x}$$

$$\frac{du}{dx} - \frac{1}{x}u = -\frac{e^{2x}}{x}.$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x -\frac{1}{s}\,ds\right) = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Proceed with the multiplication.

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -\frac{e^{2x}}{x^2}$$

The left side can be written as $d/dx(Iu)$ by the chain rule.

$$\frac{d}{dx}\left(\frac{u}{x}\right) = -\frac{e^{2x}}{x^2}$$

Integrate both sides with respect to x.

$$\frac{u}{x} = -\int^x \frac{e^{2s}}{s^2}\,ds + C$$

The lower limit of integration is arbitrary since C is as well; C will be adjusted to account for whatever choice we make. Here it will be set to 1 because of the given boundary condition.

$$\frac{u}{x} = -\int_1^x \frac{e^{2s}}{s^2}\,ds + C$$

Now that the integration is done, change back to y.

$$\frac{1}{xy} = -\int_1^x \frac{e^{2s}}{s^2}\,ds + C$$

Apply the boundary condition $y(1) = 2$ to determine C.

$$\frac{1}{(1)(2)} = C \quad \rightarrow \quad C = \frac{1}{2}$$

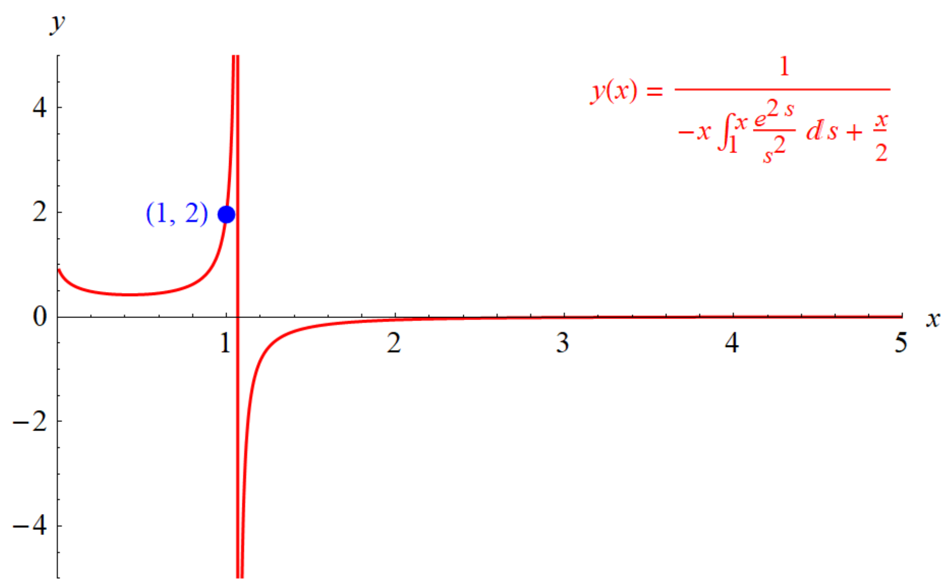
The previous equation becomes

$$\frac{1}{xy} = -\int_1^x \frac{e^{2s}}{s^2}\,ds + \frac{1}{2}$$

$$\frac{1}{y} = -x\int_1^x \frac{e^{2s}}{s^2}\,ds + \frac{x}{2}.$$

Therefore,

$$y(x) = \frac{1}{-x\displaystyle\int_1^x \frac{e^{2s}}{s^2}\,ds + \frac{x}{2}}.$$



This figure illustrates the solution to the ODE in the xy-plane. Note that the solution is only valid along the curve that passes through the point $(1, 2)$.