Problem 41

Some Special Second Order Equations. Second order equations involve the second derivative of the unknown function and have the general form $y'' = f(t, y, y')$. Usually such equations cannot be solved by methods designed for first order equations. However, there are two types of second order equations that can be transformed into first order equations by a suitable change of variable. The resulting equation can sometimes be solved by the methods presented in this chapter. Problems 36 through 51 deal with these types of equations.

Equations with the Dependent Variable Missing. For a second order differential equation of the form $y'' = f(t, y')$, the substitution $v = y'$, $v' = y''$ leads to a first order equation of the form $v' = f(t, v)$. If this equation can be solved for v, then y can be obtained by integrating $dy/dt = v$. Note that one arbitrary constant is obtained in solving the first order equation for v, and a second is introduced in the integration for y. In each of Problems 36 through 41, use this substitution to solve the given equation.

$$t^2 y'' = (y')^2, \qquad t > 0$$

Solution

Make the substitution $v = y'$.

$$t^2 y'' = v^2$$

Differentiate both sides of it with respect to t to find what y'' is in terms of this new variable.

$$v' = y''$$

Consequently, the ODE that v satisfies is

$$t^2 v' = v^2,$$

which can be solved by separating variables.

$$\frac{dv}{v^2} = \frac{dt}{t^2}$$

Integrate both sides.

$$\int \frac{dv}{v^2} = \int \frac{dt}{t^2}$$

$$-\frac{1}{v} = -\frac{1}{t} + C_1$$

Solve for v.

$$v(t) = \frac{1}{\frac{1}{t} - C_1}$$

Now that we have v, change back to y'.

$$\frac{dy}{dt} = \frac{t}{1 - C_1 t}$$

Integrate both sides with respect to t.

$$y(t) = \int^t \frac{s}{1 - C_1 s}\, ds + C_2$$

Suppose first that $C_1 \neq 0$. Make the following substitution.

$$\begin{aligned} u &= 1 - C_1 s & \quad\rightarrow\quad & \frac{1-u}{C_1} = s \\ du &= -C_1\, ds & \quad\rightarrow\quad & -\frac{du}{C_1} = ds \end{aligned}$$

As a result,

$$\begin{aligned} y(t) &= \int^{1-C_1 t} \frac{(1-u)/C_1}{u}\left(-\frac{du}{C_1}\right) + C_2 \\ &= \frac{1}{C_1^2}\int^{1-C_1 t} \frac{u-1}{u}\, du + C_2 \\ &= \frac{1}{C_1^2}\left(\int^{1-C_1 t} du - \int^{1-C_1 t} \frac{du}{u}\right) + C_2 \\ &= \frac{1}{C_1^2}\left(1 - C_1 t - \ln|1 - C_1 t|\right) + C_2 \\ &= -\frac{t}{C_1} - \frac{1}{C_1^2}\ln|1 - C_1 t| + \frac{1}{C_1^2} + C_2. \end{aligned}$$

Therefore, using the new constants, $C_3 = -C_1$ and $C_4 = 1/C_1^2 + C_2$,

$$y(t) = \frac{t}{C_3} - \frac{1}{C_3^2}\ln|1 + C_3 t| + C_4.$$

Again, this result holds for $C_3 \neq 0$. Suppose secondly that $C_1 = 0$. Then

$$\begin{aligned} y(t) &= \int^t \frac{s}{1 - C_1 s}\, ds + C_2 \\ &= \int^t s\, ds + C_2 \\ &= \frac{t^2}{2} + C_2. \end{aligned}$$

Notice also that $y(t) = C_5$ satisfies the ODE.