Problem 45

Some Special Second Order Equations. Second order equations involve the second derivative of the unknown function and have the general form $y'' = f(t, y, y')$. Usually such equations cannot be solved by methods designed for first order equations. However, there are two types of second order equations that can be transformed into first order equations by a suitable change of variable. The resulting equation can sometimes be solved by the methods presented in this chapter. Problems 36 through 51 deal with these types of equations.

Equations with the Independent Variable Missing. Consider second order differential equations of the form $y'' = f(y, y')$, in which the independent variable t does not appear explicitly. If we let $v = y'$, then we obtain $dv/dt = f(y, v)$. Since the right side of this equation depends on y and v, rather than on t and v, this equation contains too many variables. However, if we think of y as the independent variable, then by the chain rule, $dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$. Hence the original differential equation can be written as $v(dv/dy) = f(y, v)$. Provided that this first order equation can be solved, we obtain v as a function of y. A relation between y and t results from solving $dy/dt = v(y)$, which is a separable equation. Again, there are two arbitrary constants in the final result. In each of Problems 42 through 47, use this method to solve the given differential equation.

$$2y^2y'' + 2y(y')^2 = 1$$

Solution

Make the substitution $v = y'$. Then $v' = y''$, but v' can be written as

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$$

by the chain rule. Consequently, the ODE that v satisfies is

$$2y^2\left(\frac{dv}{dy}v\right) + 2yv^2 = 1.$$

Write it as $M(y, v) + N(y, v)(dv/dy) = 0$.

$$(2yv^2 - 1) + 2y^2v\frac{dv}{dy} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial v}(2yv^2 - 1) = \frac{\partial}{\partial y}(2y^2v) = 4yv.$$

That means there exists a potential function $\psi = \psi(y, v)$ that satisfies

$$\frac{\partial \psi}{\partial y} = 2yv^2 - 1 \tag{2}$$

$$\frac{\partial \psi}{\partial v} = 2y^2v. \tag{3}$$

Integrate both sides of equation (2) partially with respect to y to get ψ.

$$\psi(y, v) = y^2v^2 - y + f(v)$$

Here $f(v)$ is an arbitrary function of v. Differentiate both sides with respect to v.

$$\psi_v(y,v) = 2y^2v + f'(v)$$

Comparing this to equation (3), we see that

$$f'(v) = 0 \quad \rightarrow \quad f(v) = 0.$$

As a result, a potential function is

$$\psi(y,v) = y^2v^2 - y.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial v}\frac{dv}{dy} = 0. \tag{4}$$

Recall that the differential of $\psi(y,v)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial y}\,dy + \frac{\partial \psi}{\partial v}\,dv.$$

Dividing both sides by dy, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dy} = \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial v}\frac{dv}{dy}$$

With it, equation (4) becomes

$$\frac{d\psi}{dy} = 0.$$

Integrate both sides with respect to y.

$$\psi(y,v) = C_1$$

$$y^2v^2 - y = C_1$$

Solve for v.

$$v^2 = \frac{C_1 + y}{y^2}$$

$$v(y) = \pm\frac{\sqrt{C_1 + y}}{y}$$

Change back to y'.

$$\frac{dy}{dt} = \pm\frac{\sqrt{C_1 + y}}{y}$$

Separate variables.

$$\frac{y}{\sqrt{C_1 + y}}\,dy = \pm dt$$

Integrate both sides.

$$\int^y \frac{s}{\sqrt{C_1 + s}}\,ds = \pm t + C_2$$

Make the following substitution.

$$w = C_1 + s \quad \rightarrow \quad s = w - C_1$$
$$dw = ds$$

Then

$$\int^{C_1+y} \frac{w - C_1}{\sqrt{w}}\,dw = \pm t + C_2$$

$$\int^{C_1+y} w^{1/2}\,dw - C_1 \int^{C_1+y} w^{-1/2}\,dw = \pm t + C_2$$

$$\frac{2}{3}(C_1 + y)^{3/2} - 2C_1(C_1 + y)^{1/2} = \pm t + C_2$$

$$\frac{2}{3}(C_1 + y)^{1/2}[(C_1 + y) - 3C_1] = \pm t + C_2.$$

Therefore,

$$\frac{2}{3}(C_1 + y)^{1/2}(y - 2C_1) = \pm t + C_2.$$