Problem 47

Some Special Second Order Equations. Second order equations involve the second derivative of the unknown function and have the general form $y'' = f(t, y, y')$. Usually such equations cannot be solved by methods designed for first order equations. However, there are two types of second order equations that can be transformed into first order equations by a suitable change of variable. The resulting equation can sometimes be solved by the methods presented in this chapter. Problems 36 through 51 deal with these types of equations.

Equations with the Independent Variable Missing. Consider second order differential equations of the form $y'' = f(y, y')$, in which the independent variable t does not appear explicitly. If we let $v = y'$, then we obtain $dv/dt = f(y, v)$. Since the right side of this equation depends on y and v, rather than on t and v, this equation contains too many variables. However, if we think of y as the independent variable, then by the chain rule, $dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$. Hence the original differential equation can be written as $v(dv/dy) = f(y, v)$. Provided that this first order equation can be solved, we obtain v as a function of y. A relation between y and t results from solving $dy/dt = v(y)$, which is a separable equation. Again, there are two arbitrary constants in the final result. In each of Problems 42 through 47, use this method to solve the given differential equation.

$$y'' + (y')^2 = 2e^{-y}$$

Hint: In Problem 47 the transformed equation is a Bernoulli equation. See Problem 27 in Section 2.4.

Solution

Make the substitution $v = y'$. Then $v' = y''$, but v' can be written as

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v = \frac{1}{2}(2v)\frac{dv}{dy} = \frac{1}{2}\frac{d}{dy}(v^2)$$

by the chain rule. Consequently, the ODE that v satisfies is

$$\frac{1}{2}\frac{d}{dy}(v^2) + v^2 = 2e^{-y}.$$

Multiply both sides by 2

$$\frac{d}{dy}(v^2) + 2v^2 = 4e^{-y}$$

and make another substitution $u = v^2$.

$$\frac{du}{dy} + 2u = 4e^{-y}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^y 2\,ds\right) = e^{2y}$$

Proceed with the multiplication.

$$e^{2y}\frac{du}{dy} + 2e^{2y}u = 4e^{y}$$

The left side can be written as $d/dy(Iu)$ by the chain rule.

$$\frac{d}{dy}(e^{2y}u) = 4e^y$$

Integrate both sides with respect to y.

$$e^{2y}u = 4e^y + C_1$$

Divide both sides by e^{2y}.

$$u(y) = \frac{4e^y + C_1}{e^{2y}}$$

Change back to v^2.

$$v^2 = \frac{4e^y + C_1}{e^{2y}}$$

$$v(y) = \pm\frac{\sqrt{4e^y + C_1}}{e^y}$$

Change back to y'.

$$\frac{dy}{dt} = \pm\frac{\sqrt{4e^y + C_1}}{e^y}$$

Separate variables.

$$\frac{e^y}{\sqrt{4e^y + C_1}}\,dy = \pm dt$$

Integrate both sides.

$$\int^y \frac{e^s}{\sqrt{4e^s + C_1}}\,ds = \pm t + C_2$$

Make the following substitution.

$$w = 4e^s + C_1$$

$$dw = 4e^s\,ds \quad \to \quad \frac{dw}{4} = e^s\,ds$$

As a result,

$$\int^{4e^y + C_1} \frac{1}{\sqrt{w}}\frac{dw}{4} = \pm t + C_2$$

$$\frac{1}{4}\int^{4e^y + C_1} w^{-1/2}\,dw = \pm t + C_2$$

$$\frac{1}{4}(2w^{1/2})\bigg|^{4e^y + C_1} = \pm t + C_2$$

$$\frac{1}{2}(4e^y + C_1)^{1/2} = \pm t + C_2.$$

Multiply both sides by 2.

$$(4e^y + C_1)^{1/2} = \pm 2t + 2C_2$$

Square both sides.

$$4e^y + C_1 = (\pm 2t + 2C_2)^2$$

$$4e^y + C_1 = 4(\pm t + C_2)^2$$

$$4e^y + C_1 = 4(t \pm C_2)^2$$
$$4e^y = 4(t \pm C_2)^2 - C_1$$

Divide both sides by 4.

$$e^y = (t \pm C_2)^2 - \frac{C_1}{4}$$

Use C_3 for $\pm C_2$ and C_4 for $-C_1/4$.

$$e^y = (t + C_3)^2 + C_4$$

Therefore, taking the natural logarithm of both sides,

$$y(t) = \ln[(t + C_3)^2 + C_4].$$