Problem 48

In each of Problems 48 through 51, solve the given initial value problem using the methods of Problems 36 through 47.

$$y'y'' = 2, \qquad y(0) = 1, \quad y'(0) = 2$$

Solution

Rewrite the left side of the ODE by using the chain rule.

$$\frac{1}{2}(2y')y'' = 2$$

$$\frac{1}{2}\frac{d}{dt}[(y')^2] = 2$$

Multiply both sides by 2.

$$\frac{d}{dt}[(y')^2] = 4$$

Integrate both sides with respect to t.

$$(y')^2 = 4t + C_1$$

Apply the second initial condition $y'(0) = 2$ here to determine C_1.

$$(2)^2 = C_1 \quad \rightarrow \quad C_1 = 4$$

So the previous equation becomes

$$(y')^2 = 4t + 4.$$

Take the square root of both sides.

$$\frac{dy}{dt} = \pm\sqrt{4t+4}$$

The plus sign is chosen so that the second initial condition remains satisfied.

$$\frac{dy}{dt} = \sqrt{4t+4}$$

Integrate both sides with respect to t once more.

$$y(t) = \int^t \sqrt{4s+4}\,ds + C_2$$

Make the following substitution.

$$w = 4s + 4$$

$$dw = 4\,ds \quad \rightarrow \quad \frac{dw}{4} = ds$$

As a result,

$$\begin{aligned} y(t) &= \int^{4t+4} \sqrt{w}\,\frac{dw}{4} + C_2 \\ &= \frac{1}{4}\int^{4t+4} w^{1/2}\,dw + C_2 \\ &= \frac{1}{4}\cdot\frac{2}{3}(4t+4)^{3/2} + C_2 \\ &= \frac{1}{6}\cdot 4^{3/2}(t+1)^{3/2} + C_2 \\ &= \frac{4}{3}(t+1)^{3/2} + C_2. \end{aligned}$$

Apply the first initial condition $y(0) = 1$ to determine C_2.

$$1 = \frac{4}{3}(1)^{3/2} + C_2 \quad \rightarrow \quad C_2 = -\frac{1}{3}$$

Therefore,

$$y(t) = \frac{4}{3}(t+1)^{3/2} - \frac{1}{3}.$$