Problem 2

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = \frac{1 + \cos x}{2 - \sin y}$$

Solution

Method Using Separation of Variables

Because the ODE is of the form $y' = f(x)/g(y)$, it can be solved by separating variables.

$$(2 - \sin y)\, dy = (1 + \cos x)\, dx$$

Integrate both sides.

$$\int (2 - \sin y)\, dy = \int (1 + \cos x)\, dx$$

$$2y + \cos y = x + \sin x + C$$

Therefore,

$$2y + \cos y - x - \sin x = C.$$

Method Using an Integrating Factor

$$\frac{dy}{dx} = \frac{1+\cos x}{2-\sin y}$$

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$(2-\sin y)\frac{dy}{dx} = 1+\cos x$$

$$(-1-\cos x) + (2-\sin y)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(-1-\cos x) = \frac{\partial}{\partial x}(2-\sin y) = 0.$$

That means there exists a potential function $\psi = \psi(x,y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = -1-\cos x \tag{2}$$

$$\frac{\partial \psi}{\partial y} = 2-\sin y. \tag{3}$$

Integrate both sides of equation (2) partially with respect to x to get ψ.

$$\psi(x,y) = -x - \sin x + h(y)$$

Here $h(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x,y) = h'(y)$$

Comparing this to equation (3), we see that

$$h'(y) = 2-\sin y \quad \rightarrow \quad h(y) = 2y + \cos y.$$

As a result, a potential function is

$$\psi(x,y) = -x - \sin x + 2y + \cos y.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x,y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

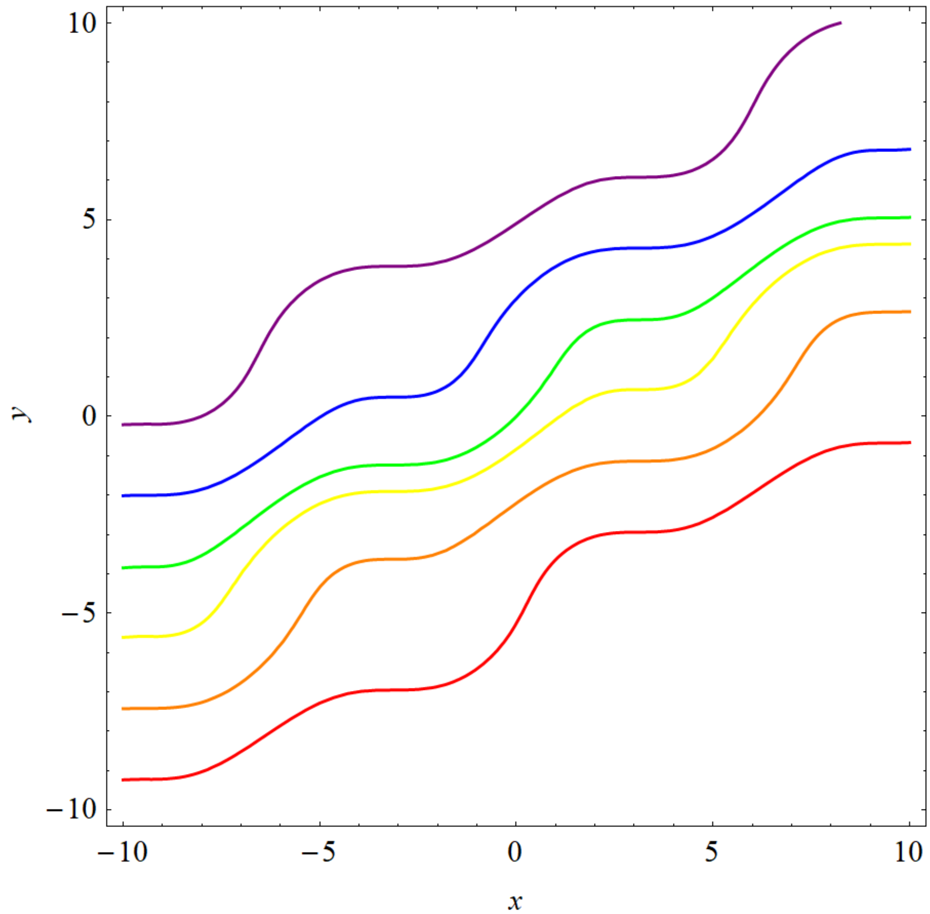
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x,y) = C_1$$

Therefore,

$$-x - \sin x + 2y + \cos y = C_1.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.