Problem 20

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$y' = e^{x+y}$$

Solution

Method Using Separation of Variables

$$\frac{dy}{dx} = e^x e^y$$

Because the ODE is of the form $y' = f(x)g(y)$, it can be solved by separating variables.

$$e^{-y}\,dy = e^x\,dx$$

Integrate both sides.

$$-e^{-y} = e^x + C$$

Solve for y.

$$e^{-y} = -e^x - C$$

$$-y = \ln(-e^x - C)$$

Therefore,

$$y(x) = -\ln(-e^x - C).$$

Method Using an Integrating Factor

$$y' = e^{x+y}$$

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$(-e^{x+y}) + \frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(-e^{x+y}) = -e^{x+y} \neq \frac{\partial}{\partial x}(1) = 0.$$

To solve it, we seek an integrating factor $\mu = \mu(x,y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(-e^{x+y}\mu) + \mu\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}(-e^{x+y}\mu) = \frac{\partial}{\partial x}(\mu).$$

Expand both sides.

$$-e^{x+y}\mu - e^{x+y}\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on y: $\mu = \mu(y)$.

$$-e^{x+y}\mu - e^{x+y}\frac{d\mu}{dy} = 0$$

$$\mu + \frac{d\mu}{dy} = 0$$

$$\frac{d\mu}{dy} = -\mu$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = -dy$$

Integrate both sides.

$$\ln \mu = -y + C_1$$

Exponentiate both sides.

$$\begin{aligned} \mu &= e^{-y+C_1} \\ &= e^{-y}e^{C_1} \end{aligned}$$

Taking e^{C_1} to be 1, an integrating factor is

$$\mu = e^{-y}.$$

Multiply both sides of equation (1) by e^{-y}.

$$(-e^{x}) + e^{-y}\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = -e^x \tag{3}$$

$$\frac{\partial \psi}{\partial y} = e^{-y}. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = -e^{-y} + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = -e^x \quad \rightarrow \quad f(x) = -e^x.$$

Consequently, a potential function is

$$\psi(x, y) = -e^{-y} - e^x.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

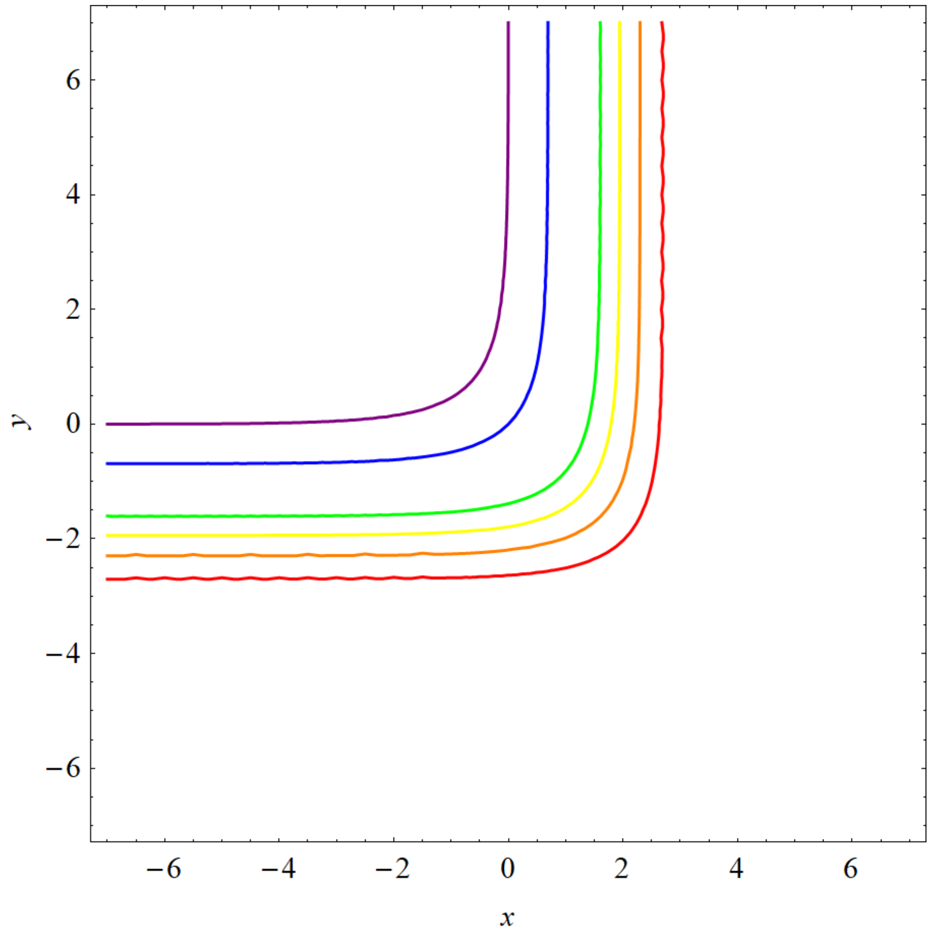
$$\psi(x, y) = C_2$$

Therefore,

$$-e^{-y} - e^x = C_2,$$

or solving for y explicitly,

$$y(x) = -\ln(-e^x - C_2).$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -15$, $C = -10$, $C = -7$, $C = -5$, $C = -2$, and $C = -1$, respectively.