Problem 21

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} + \frac{2y^2 + 6xy - 4}{3x^2 + 4xy + 3y^2} = 0$$

Solution

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$(2y^2 + 6xy - 4) + (3x^2 + 4xy + 3y^2)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(2y^2 + 6xy - 4) = \frac{\partial}{\partial x}(3x^2 + 4xy + 3y^2) = 6x + 4y.$$

That means there exists a potential function $\psi = \psi(x, y)$ which satisfies

$$\frac{\partial \psi}{\partial x} = 2y^2 + 6xy - 4 \tag{2}$$

$$\frac{\partial \psi}{\partial y} = 3x^2 + 4xy + 3y^2. \tag{3}$$

Integrate both sides of equation (3) partially with respect to y to get ψ.

$$\psi(x, y) = 3x^2y + 2xy^2 + y^3 + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = 6xy + 2y^2 + f'(x)$$

Comparing this to equation (2), we see that

$$f'(x) = -4 \quad \rightarrow \quad f(x) = -4x.$$

Consequently, a potential function is

$$\psi(x, y) = 3x^2y + 2xy^2 + y^3 - 4x.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

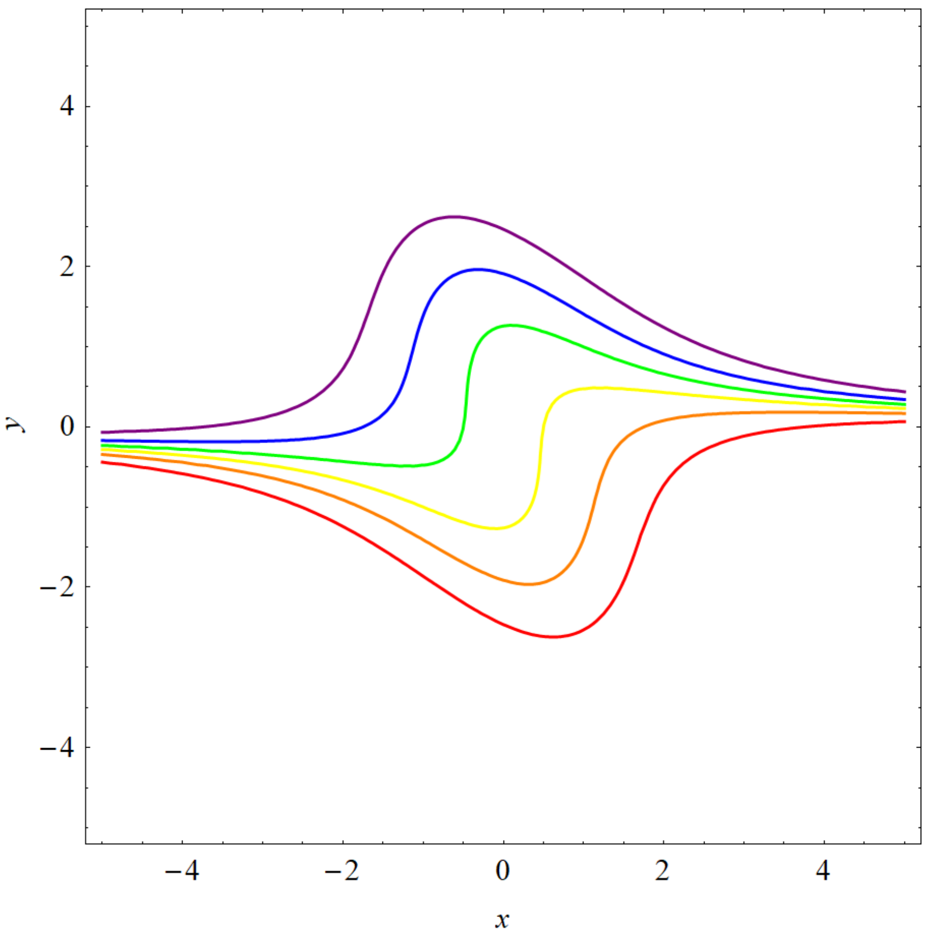
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$3x^2y + 2xy^2 + y^3 - 4x = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -15$, $C = -7$, $C = -2$, $C = 2$, $C = 7$, and $C = 15$, respectively.