Problem 26

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$xy' = y + xe^{y/x}$$

Solution

Divide both sides by x.

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

Make the substitution $u = y/x$.

$$\frac{dy}{dx} = u + e^u$$

Differentiate both sides of the substitution with respect to x to find what dy/dx is in terms of this new variable.

$$\frac{du}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} \quad \rightarrow \quad x\frac{du}{dx} = \frac{dy}{dx} - \frac{y}{x} \quad \rightarrow \quad \frac{dy}{dx} = x\frac{du}{dx} + \frac{y}{x} = x\frac{du}{dx} + u$$

Consequently, the ODE that u satisfies is

$$x\frac{du}{dx} + u = u + e^u$$

$$x\frac{du}{dx} = e^u,$$

which can be solved by separating variables.

$$e^{-u}\,du = \frac{dx}{x}$$

Integrate both sides.

$$\int e^{-u}\,du = \int \frac{dx}{x}$$

$$-e^{-u} = \ln|x| + C$$

Now that the integration is done, change back to y.

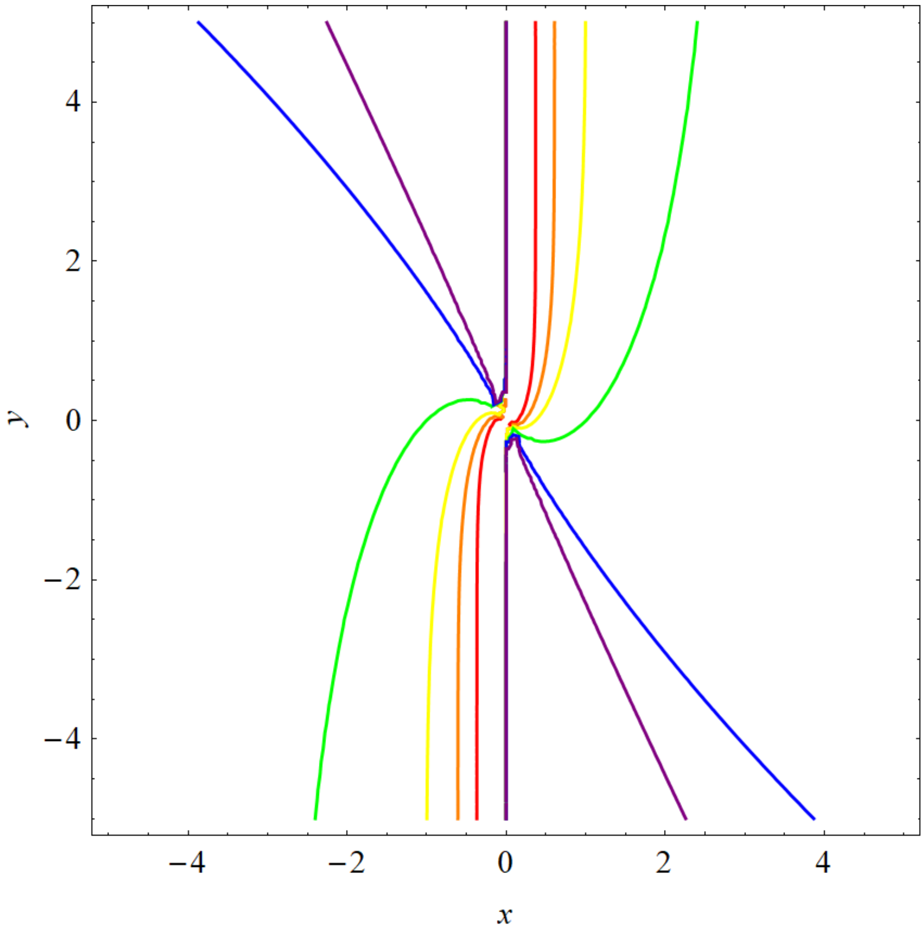
$$-e^{-y/x} = \ln|x| + C$$

Therefore,

$$e^{-y/x} + \ln|x| = C_1,$$

or solving for y explicitly,

$$y(x) = -x\ln(C_1 - \ln|x|).$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -1$, $C = -0.5$, $C = 0$, $C = 1$, $C = 5$, and $C = 10$, respectively.