Problem 31

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = -\frac{3x^2y + y^2}{2x^3 + 3xy}, \qquad y(1) = -2$$

Solution

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$(2x^3 + 3xy)\frac{dy}{dx} = -(3x^2y + y^2)$$

$$(3x^2y + y^2) + (2x^3 + 3xy)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(3x^2y + y^2) = 3x^2 + 2y \neq \frac{\partial}{\partial x}(2x^3 + 3xy) = 6x^2 + 3y.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(3x^2y + y^2)\mu + \mu(2x^3 + 3xy)\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(3x^2y + y^2)\mu] = \frac{\partial}{\partial x}[\mu(2x^3 + 3xy)].$$

Expand both sides.

$$(3x^2 + 2y)\mu + (3x^2y + y^2)\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}(2x^3 + 3xy) + \mu(6x^2 + 3y)$$

Assume that μ is only dependent on y: $\mu = \mu(y)$.

$$(3x^2 + 2y)\mu + (3x^2y + y^2)\frac{d\mu}{dy} = \mu(6x^2 + 3y)$$

$$(3x^2y + y^2)\frac{d\mu}{dy} = \mu(3x^2 + y)$$

$$(3x^2 + y)y\frac{d\mu}{dy} = \mu(3x^2 + y)$$

$$y\frac{d\mu}{dy} = \mu$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = \frac{dy}{y}$$

Integrate both sides.

$$\ln \mu = \ln y + C$$

Exponentiate both sides.

$$\mu = ye^C$$

Taking e^C to be 1, an integrating factor is

$$\mu = y.$$

Multiply both sides of equation (1) by y.

$$(3x^2y^2 + y^3) + (2x^3y + 3xy^2)\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial\psi}{\partial x} = 3x^2y^2 + y^3 \tag{3}$$

$$\frac{\partial\psi}{\partial y} = 2x^3y + 3xy^2. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = x^3y^2 + xy^3 + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = 3x^2y^2 + y^3 + f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = 0 \quad \rightarrow \quad f(x) = 0.$$

Consequently, a potential function is

$$\psi(x, y) = x^3y^2 + xy^3.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_1$$

The general solution is then

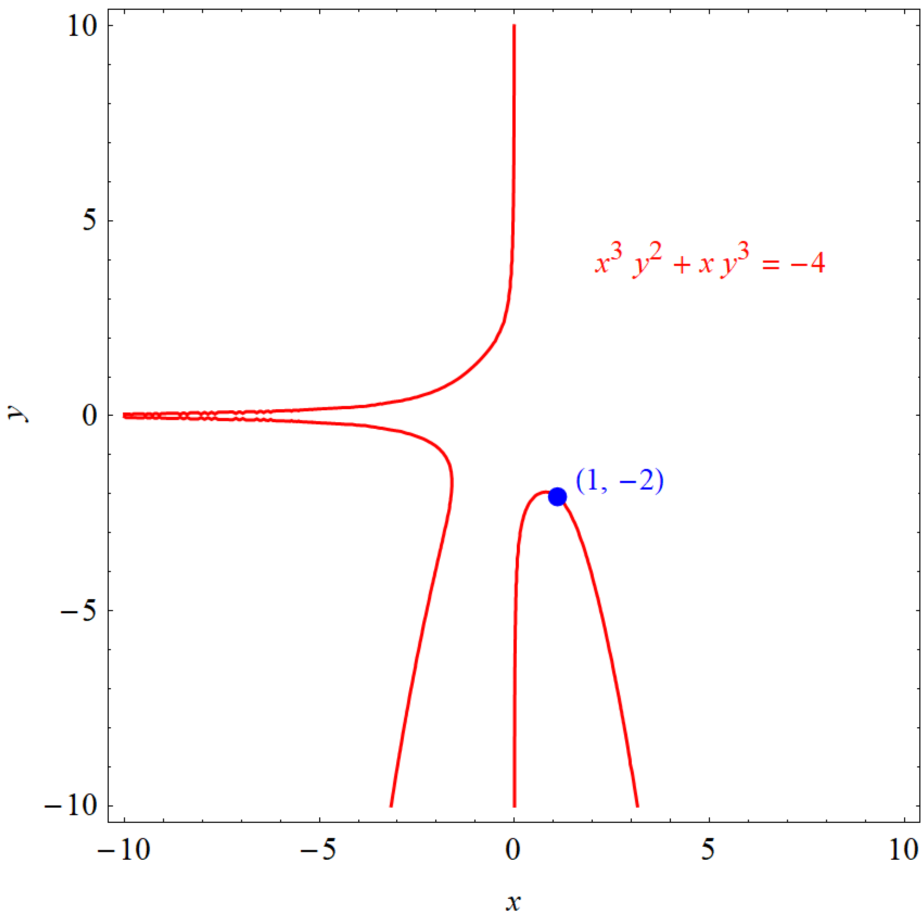
$$x^3y^2 + xy^3 = C_1.$$

Apply the boundary condition $y(1) = -2$ to determine C_1.

$$1^3(-2)^2 + (1)(-2)^3 = C_1 \quad \rightarrow \quad C_1 = -4$$

Therefore,

$$x^3y^2 + xy^3 = -4.$$



This figure illustrates the solution to the ODE in the xy-plane. Note that it's only valid along the curve that passes through the point $(1, -2)$.