Problem 34

Using the method of Problem 33 and the given particular solution, solve each of the following Riccati equations:

(a) $y' = 1 + t^2 - 2ty + y^2; \qquad y_1(t) = t$ (b) $y' = -\frac{1}{t^2} - \frac{y}{t} + y^2; \qquad y_1(t) = \frac{1}{t}$

(c) $\frac{dy}{dt} = \frac{2\cos^2 t - \sin^2 t + y^2}{2\cos t}; \qquad y_1(t) = \sin t$

Solution

Part (a)

Make the substitution

$$y = t + \frac{1}{v(t)}.$$

Differentiate both sides of the substitution with respect to t to find what dy/dt is in terms of this new variable.

$$\frac{dy}{dt} = 1 - \frac{1}{[v(t)]^2}\frac{dv}{dt}$$

Substitute these previous two expressions into the ODE.

$$1 - \frac{1}{[v(t)]^2}\frac{dv}{dt} = 1 + t^2 - 2t\left[t + \frac{1}{v(t)}\right] + \left[t + \frac{1}{v(t)}\right]^2$$

Cancel 1 from both sides and expand the right side.

$$-\frac{1}{v^2}\frac{dv}{dt} = t^2 - 2t^2 - \frac{2t}{v} + t^2 + \frac{2t}{v} + \frac{1}{v^2}$$

$$-\frac{1}{v^2}\frac{dv}{dt} = \frac{1}{v^2}$$

Multiply both sides by $-v^2$.

$$\frac{dv}{dt} = -1$$

Integrate both sides with respect to t.

$$v(t) = -t + C$$

Therefore, the general solution is

$$y(t) = t + \frac{1}{-t + C}.$$

Part (b)

$$y' = -\frac{1}{t^2} - \frac{y}{t} + y^2; \qquad y_1(t) = \frac{1}{t}$$

Make the substitution

$$y = \frac{1}{t} + \frac{1}{v(t)}.$$

Differentiate both sides of the substitution with respect to t to find what dy/dt is in terms of this new variable.

$$\frac{dy}{dt} = -\frac{1}{t^2} - \frac{1}{[v(t)]^2}\frac{dv}{dt}$$

Substitute these previous two expressions into the ODE.

$$-\frac{1}{t^2} - \frac{1}{[v(t)]^2}\frac{dv}{dt} = -\frac{1}{t^2} - \frac{1}{t}\left[\frac{1}{t} + \frac{1}{v(t)}\right] + \left[\frac{1}{t} + \frac{1}{v(t)}\right]^2$$

Cancel $-1/t^2$ from both sides and expand the right side.

$$\begin{aligned} -\frac{1}{v^2}\frac{dv}{dt} &= -\frac{1}{t^2} - \frac{1}{tv} + \frac{1}{t^2} + \frac{2}{tv} + \frac{1}{v^2} \\ &= \frac{1}{tv} + \frac{1}{v^2} \end{aligned}$$

Multiply both sides by $-v^2$.

$$\begin{aligned} \frac{dv}{dt} &= -\frac{1}{t}v - 1 \\ \frac{dv}{dt} &+ \frac{1}{t}v = -1 \end{aligned}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{1}{s}\,ds\right) = e^{\ln t} = t$$

Proceed with the multiplication.

$$t\frac{dv}{dt} + v = -t$$

The left side can be written as $d/dt(Iv)$ by the chain rule.

$$\frac{d}{dt}(tv) = -t$$

Integrate both sides with respect to t.

$$tv = -\frac{t^2}{2} + C$$

Divide both sides by t.

$$v(t) = \frac{-\frac{t^2}{2} + C}{t}$$

Therefore, the general solution is

$$y(t) = \frac{1}{t} + \frac{t}{-\frac{t^2}{2} + C}.$$

Part (c)

$$\frac{dy}{dt} = \frac{2\cos^2 t - \sin^2 t + y^2}{2\cos t}; \qquad y_1(t) = \sin t$$

Make the substitution

$$y = \sin t + \frac{1}{v(t)}.$$

Differentiate both sides of the substitution with respect to t to find what dy/dt is in terms of this new variable.

$$\frac{dy}{dt} = \cos t - \frac{1}{[v(t)]^2}\frac{dv}{dt}$$

Substitute these previous two expressions into the ODE.

$$\begin{aligned}
\cos t - \frac{1}{[v(t)]^2}\frac{dv}{dt} &= \frac{2\cos^2 t - \sin^2 t + \left[\sin t + \frac{1}{v(t)}\right]^2}{2\cos t} \\
&= \frac{2\cos^2 t - \sin^2 t + \sin^2 t + 2\frac{\sin t}{v(t)} + \frac{1}{[v(t)]^2}}{2\cos t} \\
&= \frac{2\cos^2 t + 2\frac{\sin t}{v(t)} + \frac{1}{[v(t)]^2}}{2\cos t} \\
&= \cos t + \frac{\sin t}{\cos t}\frac{1}{v(t)} + \frac{1}{2\cos t}\frac{1}{[v(t)]^2}
\end{aligned}$$

Cancel $\cos t$ from both sides.

$$-\frac{1}{v^2}\frac{dv}{dt} = \frac{\sin t}{\cos t}\frac{1}{v} + \frac{1}{2\cos t}\frac{1}{v^2}$$

Multiply both sides by $-v^2$.

$$\begin{aligned}
\frac{dv}{dt} &= -\frac{\sin t}{\cos t}v - \frac{1}{2\cos t} \\
\frac{dv}{dt} + \frac{\sin t}{\cos t}v &= -\frac{1}{2\cos t}
\end{aligned}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{\sin s}{\cos s}\,ds\right) = e^{-\ln\cos t} = e^{\ln\frac{1}{\cos t}} = \frac{1}{\cos t}$$

Proceed with the multiplication.

$$\frac{1}{\cos t}\frac{dv}{dt} + \frac{\sin t}{\cos^2 t}v = -\frac{1}{2\cos^2 t}$$

The left side can be written as $d/dt(Iv)$ by the chain rule.

$$\frac{d}{dt}\left(\frac{v}{\cos t}\right) = -\frac{1}{2\cos^2 t}$$

Integrate both sides with respect to t.

$$\begin{aligned}
\frac{v}{\cos t} &= \int -\frac{1}{2\cos^2 t}\,dt \\
&= -\frac{1}{2}\int \sec^2 t\,dt \\
&= -\frac{1}{2}\tan t + C \\
&= -\frac{1}{2}\frac{\sin t}{\cos t} + C
\end{aligned}$$

Multiply both sides by $\cos t$.

$$v(t) = -\frac{1}{2}\sin t + C\cos t$$

Therefore, the general solution is

$$y(t) = \sin t + \frac{1}{-\frac{1}{2}\sin t + C\cos t}.$$