Problem 35

The propagation of a single action in a large population (for example, drivers turning on headlights at sunset) often depends partly on external circumstances (gathering darkness) and partly on a tendency to imitate others who have already performed the action in question. In this case the proportion $y(t)$ of people who have performed the action can be described[24] by the equation

$$dy/dt = (1-y)[x(t) + by], \tag{i}$$

where $x(t)$ measures the external stimulus and b is the imitation coefficient.

(a) Observe that Eq. (i) is a Riccati equation and that $y_1(t) = 1$ is one solution. Use the transformation suggested in Problem 33, and find the linear equation satisfied by $v(t)$.

(b) Find $v(t)$ in the case that $x(t) = at$, where a is a constant. Leave your answer in the form of an integral.

Solution

Make the substitution

$$y = 1 + \frac{1}{v(t)}.$$

Differentiate both sides of the substitution with respect to t to find what dy/dt is in terms of this new variable.

$$\frac{dy}{dt} = -\frac{1}{[v(t)]^2}\frac{dv}{dt}$$

Substitute these previous two expressions into the ODE.

$$-\frac{1}{[v(t)]^2}\frac{dv}{dt} = \left\{1 - \left[1 + \frac{1}{v(t)}\right]\right\}\left\{x(t) + b\left[1 + \frac{1}{v(t)}\right]\right\}$$

$$-\frac{1}{v^2}\frac{dv}{dt} = -\frac{1}{v}\left(x + b + \frac{b}{v}\right)$$

Multiply both sides by $-v^2$.

$$\frac{dv}{dt} = (x+b)v + b$$

Assuming that $x(t) = at$ and bringing the term with v to the left side, we get

$$\frac{dv}{dt} - (at+b)v = b.$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t -(as+b)\,ds\right] = \exp\left(-\frac{1}{2}at^2 - bt\right)$$

[24] See Anatol Rapoport, "Contribution to the Mathematical Theory of Mass Behavior: I. The Propagation of Single Acts," *Bulletin of Mathematical Biophysics 14* (1952), pp. 159-169, and John Z. Hearon, "Note on the Theory of Mass Behavior," *Bulletin of Mathematical Biophysics 17* (1955), pp. 7-13.

Proceed with the multiplication.

$$\exp\left(-\frac{1}{2}at^2 - bt\right)\frac{dv}{dt} - (at+b)\exp\left(-\frac{1}{2}at^2 - bt\right)v = b\exp\left(-\frac{1}{2}at^2 - bt\right)$$

The left side can be written as $d/dt(Iv)$ by the chain rule.

$$\frac{d}{dt}\left[\exp\left(-\frac{1}{2}at^2 - bt\right)v\right] = b\exp\left(-\frac{1}{2}at^2 - bt\right)$$

Integrate both sides with respect to t.

$$\exp\left(-\frac{1}{2}at^2 - bt\right)v = \int^t b\exp\left(-\frac{1}{2}as^2 - bs\right)ds + C$$

Multiply both sides by $e^{at^2/2+bt}$ to solve for v.

$$\begin{aligned} v(t) &= \exp\left(\frac{1}{2}at^2 + bt\right)\int^t b\exp\left(-\frac{1}{2}as^2 - bs\right)ds + C\exp\left(\frac{1}{2}at^2 + bt\right) \\ &= \int^t b\exp\left(\frac{1}{2}at^2 + bt\right)\exp\left(-\frac{1}{2}as^2 - bs\right)ds + C\exp\left(\frac{1}{2}at^2 + bt\right) \\ &= b\int^t \exp\left[\frac{1}{2}a(t^2 - s^2) + b(t-s)\right]ds + C\exp\left(\frac{1}{2}at^2 + bt\right) \end{aligned}$$

Therefore,

$$y(t) = 1 + \frac{1}{b\displaystyle\int^t \exp\left[\frac{1}{2}a(t^2 - s^2) + b(t-s)\right]ds + C\exp\left(\frac{1}{2}at^2 + bt\right)}.$$