Problem 49

In each of Problems 48 through 51, solve the given initial value problem using the methods of Problems 36 through 47.

$$y'' - 3y^2 = 0, \qquad y(0) = 2, \quad y'(0) = 4$$

Solution

Make the substitution $v = y'$. Then $v' = y''$, but v' can be written as

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$$

by the chain rule. Consequently, the ODE that v satisfies is

$$\frac{dv}{dy}v - 3y^2 = 0,$$

which can be solved by separating variables.

$$\frac{dv}{dy}v = 3y^2$$

$$v\,dv = 3y^2\,dy$$

Integrate both sides.

$$\frac{1}{2}v^2 = y^3 + C_1$$

Multiply both sides 2.

$$v^2 = 2y^3 + 2C_1$$

$$v(y) = \pm\sqrt{2y^3 + 2C_1}$$

Change back to y'.

$$\frac{dy}{dt} = \pm\sqrt{2y^3 + 2C_1}$$

Use the two initial conditions, $y(0) = 2$ and $y'(0) = 4$, to determine C_1.

$$4 = \pm\sqrt{2(2)^3 + 2C_1} \quad \rightarrow \quad 2C_1 = 0$$

Note that the plus sign has to be chosen to get 4 on the right side. The previous equation then becomes

$$\frac{dy}{dt} = \sqrt{2y^3}.$$

Separate variables once more.

$$y^{-3/2}\,dy = \sqrt{2}\,dt$$

Integrate both sides.

$$-2y^{-1/2} = \sqrt{2}t + C_2$$

Use the first initial condition $y(0) = 2$ to determine C_2.

$$-2(2)^{-1/2} = C_2 \quad \rightarrow \quad C_2 = -\sqrt{2}$$

So then

$$-2y^{-1/2} = \sqrt{2}t - \sqrt{2}.$$

Solve for y.

$$\begin{aligned} y^{-1/2} &= -\frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}} \\ &= \frac{-t+1}{\sqrt{2}} \end{aligned}$$

$$y^{1/2} = \frac{\sqrt{2}}{-t+1}$$

Therefore, squaring both sides,

$$y(t) = \frac{2}{(1-t)^2}.$$