Problem 50

In each of Problems 48 through 51, solve the given initial value problem using the methods of Problems 36 through 47.

$$(1+t^2)y'' + 2ty' + 3t^{-2} = 0, \qquad y(1) = 2, \quad y'(1) = -1$$

Solution

Bring the third term to the right side and divide both sides by $1+t^2$.

$$y'' + \frac{2t}{1+t^2}y' = -\frac{3}{t^2(1+t^2)}$$

This is a first-order linear inhomogeneous ODE for y', so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{2s}{1+s^2}\,ds\right) = e^{\ln(1+t^2)} = 1+t^2$$

Proceed with the multiplication.

$$(1+t^2)y'' + 2ty' = -\frac{3}{t^2}$$

The left side can be written as $d/dt(Iy')$ by the chain rule.

$$\frac{d}{dt}[(1+t^2)y'] = -\frac{3}{t^2}$$

Integrate both sides with respect to t.

$$(1+t^2)y' = \frac{3}{t} + C_1$$

Apply the second initial condition $y'(1) = -1$ to determine C_1.

$$(1+1^2)(-1) = 3 + C_1 \quad \to \quad C_1 = -5$$

The previous equation then becomes

$$(1+t^2)y' = \frac{3}{t} - 5$$

Divide both sides by $1+t^2$.

$$\frac{dy}{dt} = \frac{3}{t(1+t^2)} - \frac{5}{1+t^2}$$

Integrate both sides with respect to t once more.

$$\begin{aligned} y(t) &= \int^t \frac{3}{s(1+s^2)}\,ds - 5\tan^{-1}t + C_2 \\ &= \int^t \left(\frac{3}{s} - \frac{3s}{1+s^2}\right) ds - 5\tan^{-1}t + C_2 \\ &= 3\ln t - \frac{3}{2}\ln(1+t^2) - 5\tan^{-1}t + C_2 \end{aligned}$$

Use the first initial condition $y(1) = 2$ to determine C_2 now.

$$2 = 3\ln 1 - \frac{3}{2}\ln 2 - 5\tan^{-1} 1 + C_2 \quad \rightarrow \quad C_2 = \frac{1}{4}(8 + 5\pi + 6\ln 2)$$

So then

$$\begin{aligned} y(t) &= 3\ln t - \frac{3}{2}\ln(1+t^2) - 5\tan^{-1} t + \frac{1}{4}(8 + 5\pi + 6\ln 2) \\ &= 3\left[\ln t - \frac{1}{2}\ln(1+t^2)\right] - 5\tan^{-1} t + \frac{1}{4}(8 + 5\pi + 6\ln 2) \\ &= 3\left(\ln t - \ln\sqrt{1+t^2}\right) - 5\tan^{-1} t + \frac{1}{4}(8 + 5\pi + 6\ln 2). \end{aligned}$$

Therefore,

$$y(t) = 3\ln\frac{t}{\sqrt{1+t^2}} - 5\tan^{-1} t + \frac{1}{4}(8 + 5\pi + 6\ln 2).$$