Problem 9

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = -\frac{2xy+1}{x^2+2y}$$

Solution

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$(x^2+2y)\frac{dy}{dx} = -(2xy+1)$$

$$(2xy+1) + (x^2+2y)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(2xy+1) = \frac{\partial}{\partial x}(x^2+2y) = 2x.$$

That means there exists a potential function $\psi = \psi(x,y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 2xy+1 \tag{2}$$

$$\frac{\partial \psi}{\partial y} = x^2+2y. \tag{3}$$

Integrate both sides of equation (3) partially with respect to y to get ψ.

$$\psi(x,y) = x^2y + y^2 + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x,y) = 2xy + f'(x)$$

Comparing this to equation (2), we see that

$$f'(x) = 1 \quad \rightarrow \quad f(x) = x.$$

As a result, a potential function is

$$\psi(x,y) = x^2y + y^2 + x.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x,y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

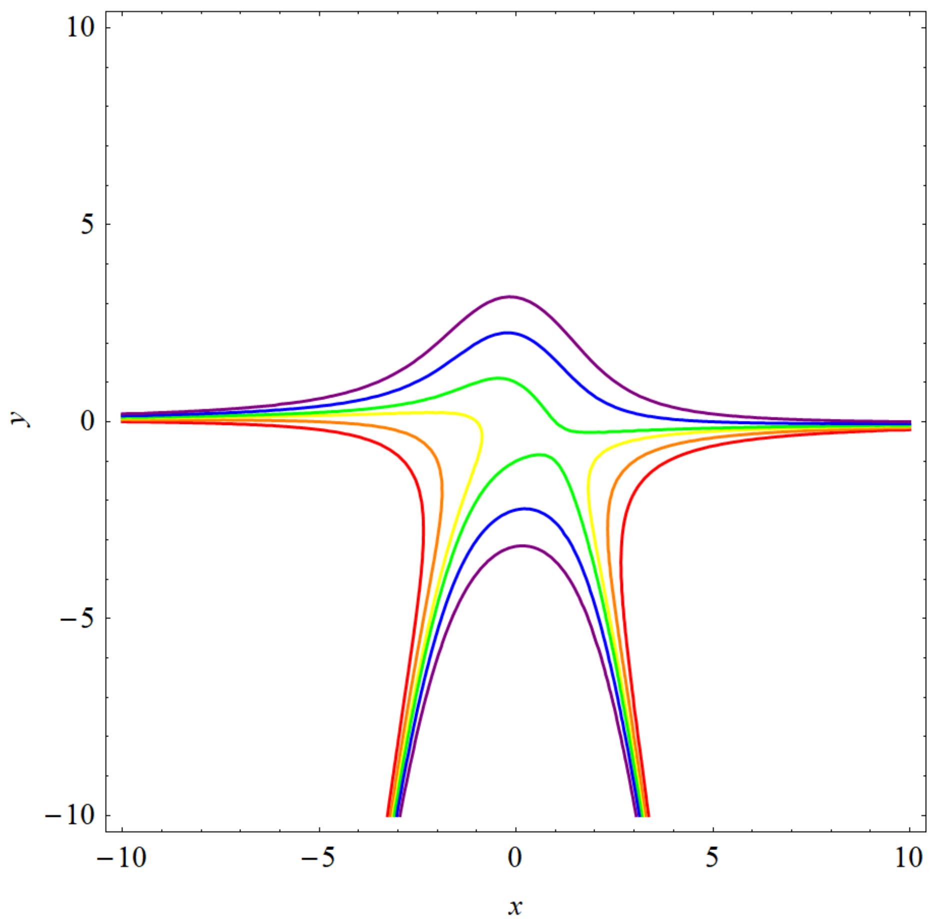
$$\psi(x, y) = C$$

Therefore,

$$x^2y + y^2 + x = C,$$

or solving for y explicitly,

$$y(x) = \frac{-x^2 \pm \sqrt{x^4 - 4(x - C)}}{2}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.