Problem 14

In each of Problems 9 through 16, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

$$2y'' + y' - 4y = 0, \qquad y(0) = 0, \quad y'(0) = 1$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$2(r^2 e^{rt}) + re^{rt} - 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$2r^2 + r - 4 = 0$$

$$r = \frac{-1 \pm \sqrt{1 + 4(4)(2)}}{2(2)} = \frac{-1 \pm \sqrt{33}}{4}$$

$$r = \left\{ \frac{-1 - \sqrt{33}}{4}, \frac{-1 + \sqrt{33}}{4} \right\}$$

Two solutions to the ODE are

$$y = \exp\left(\frac{-1 - \sqrt{33}}{4} t\right) \quad \text{and} \quad y = \exp\left(\frac{-1 + \sqrt{33}}{4} t\right),$$

so the general solution is

$$y(t) = C_1 \exp\left(\frac{-1 - \sqrt{33}}{4} t\right) + C_2 \exp\left(\frac{-1 + \sqrt{33}}{4} t\right),$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = C_1 \left(\frac{-1 - \sqrt{33}}{4}\right) \exp\left(\frac{-1 - \sqrt{33}}{4} t\right) + C_2 \left(\frac{-1 + \sqrt{33}}{4}\right) \exp\left(\frac{-1 + \sqrt{33}}{4} t\right),$$

Apply the two initial conditions now to determine C_1 and C_2.

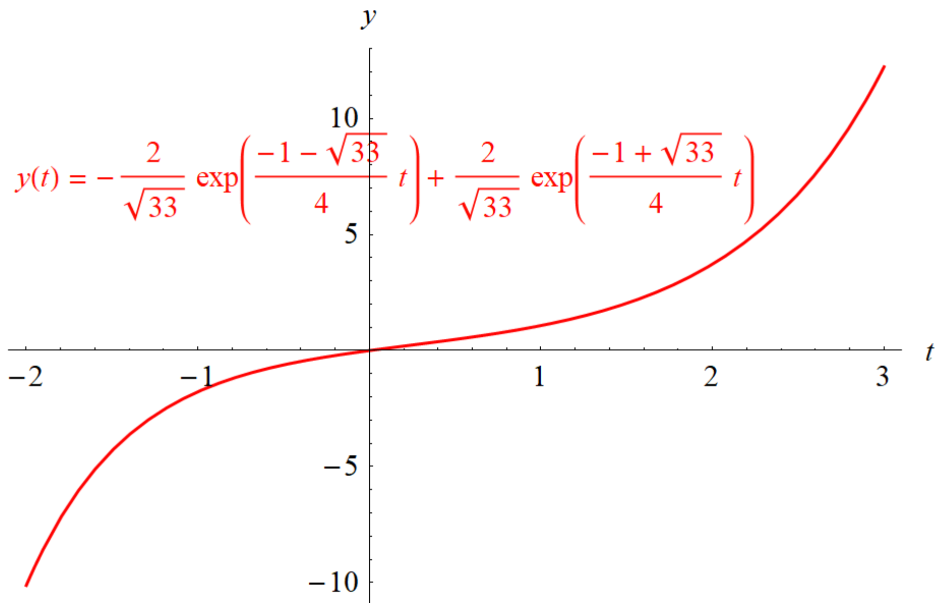
$$y(0) = C_1 + C_2 = 0$$

$$y'(0) = C_1 \left(\frac{-1 - \sqrt{33}}{4}\right) + C_2 \left(\frac{-1 + \sqrt{33}}{4}\right) = 1$$

Solving the system of equations yields $C_1 = -2/\sqrt{33}$ and $C_2 = 2/\sqrt{33}$. Therefore,

$$y(t) = -\frac{2}{\sqrt{33}} \exp\left(\frac{-1 - \sqrt{33}}{4} t\right) + \frac{2}{\sqrt{33}} \exp\left(\frac{-1 + \sqrt{33}}{4} t\right).$$

Because the coefficient of t in the second exponential function is positive, this solution diverges to ∞ as $t \to \infty$.


y
10
$y(t) = -\frac{2}{\sqrt{33}} \exp\left(\frac{-1-\sqrt{33}}{4} t\right) + \frac{2}{\sqrt{33}} \exp\left(\frac{-1+\sqrt{33}}{4} t\right)$
5
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