Problem 20

Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$2(r^2 e^{rt}) - 3(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$2r^2 - 3r + 1 = 0$$

$$(2r - 1)(r - 1) = 0$$

$$r = \left\{\frac{1}{2}, 1\right\}$$

Two solutions to the ODE are $y = e^{t/2}$ and $y = e^t$, so the general solution is

$$y(t) = C_1 e^{t/2} + C_2 e^t,$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = \frac{C_1}{2} e^{t/2} + C_2 e^t$$

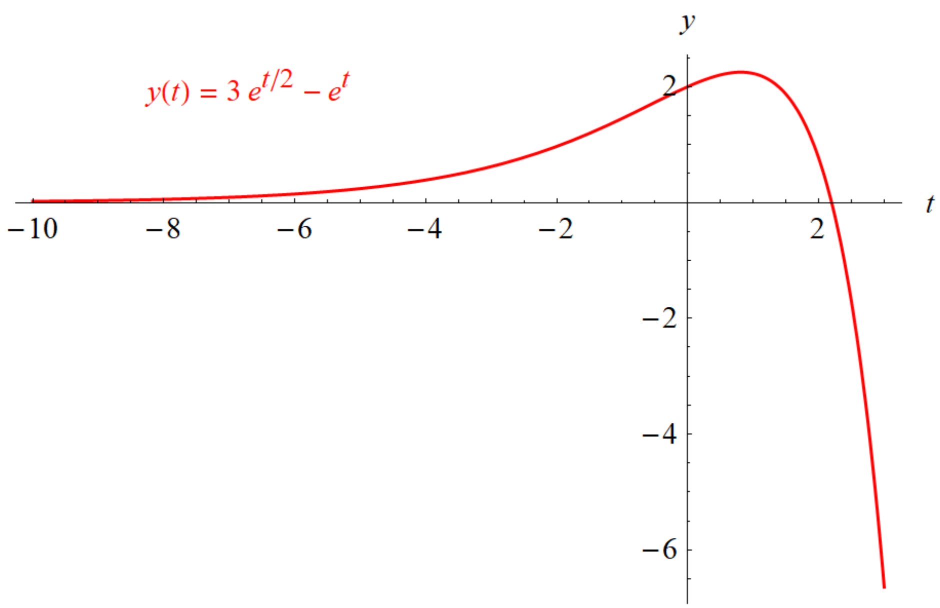
Apply the two initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 2$$

$$y'(0) = \frac{C_1}{2} + C_2 = \frac{1}{2}$$

Solving the system of equations yields $C_1 = 3$ and $C_2 = -1$. Therefore,

$$y(t) = 3e^{t/2} - e^t.$$



The maximum value is found by solving $y'(t) = 0$ for t, and the zero is found by solving $y(t) = 0$ for t.

$$y'(t) = 0 \qquad\qquad y(t) = 0$$

$$\frac{3}{2}e^{t/2} - e^t = 0 \qquad\qquad 3e^{t/2} - e^t = 0$$

Divide both sides of each equation by $e^{t/2}$.

$$\frac{3}{2} - e^{t/2} = 0 \qquad\qquad 3 - e^{t/2} = 0$$

$$e^{t/2} = \frac{3}{2} \qquad\qquad e^{t/2} = 3$$

$$\ln e^{t/2} = \ln\frac{3}{2} \qquad\qquad \ln e^{t/2} = \ln 3$$

$$\frac{t}{2} = \ln\frac{3}{2} \qquad\qquad \frac{t}{2} = \ln 3$$

$$t = 2\ln\frac{3}{2} \approx 0.811 \qquad\qquad t = 2\ln 3 \approx 2.20$$

Therefore, the maximum is

$$y\left(2\ln\frac{3}{2}\right) = \frac{9}{4},$$

and the zero is at $(2\ln 3, 0)$.