Problem 23

In each of Problems 23 and 24, determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - (2\alpha - 1)(re^{rt}) + \alpha(\alpha - 1)(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$$

$$r = \frac{(2\alpha - 1) \pm \sqrt{(2\alpha - 1)^2 - 4\alpha(\alpha - 1)}}{2} = \frac{2\alpha - 1 \pm 1}{2}$$

$$r = \{\alpha - 1, \alpha\}$$

Two solutions to the ODE are $y = e^{(\alpha-1)t}$ and $y = e^{\alpha t}$. In order for both solutions to converge to zero as $t \to \infty$, both coefficients of t in the exponents must be negative.

$$\begin{aligned} \alpha - 1 < 0 \quad &\text{and} \quad \alpha < 0 \\ \alpha < 1 \quad &\text{and} \quad \alpha < 0 \end{aligned}$$

Therefore, $y \to 0$ as $t \to \infty$ if $\alpha < 0$. On the other hand, for both solutions to diverge to ∞ as $t \to \infty$, both coefficients of t in the exponents must be positive.

$$\begin{aligned} \alpha - 1 > 0 \quad &\text{and} \quad \alpha > 0 \\ \alpha > 1 \quad &\text{and} \quad \alpha > 0 \end{aligned}$$

Therefore, $y \to \infty$ as $t \to \infty$ if $\alpha > 1$.