Problem 24

In each of Problems 23 and 24, determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

$$y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + (3 - \alpha)(re^{rt}) - 2(\alpha - 1)(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + (3 - \alpha)r - 2(\alpha - 1) = 0$$

$$r = \frac{-(3 - \alpha) \pm \sqrt{(3 - \alpha)^2 + 4 \cdot 2(\alpha - 1)}}{2} = \frac{-3 + \alpha \pm \sqrt{(\alpha + 1)^2}}{2} = \frac{-3 + \alpha \pm (\alpha + 1)}{2}$$

$$r = \{-2, \alpha - 1\}$$

Two solutions to the ODE are $y = e^{-2t}$ and $y = e^{(\alpha - 1)t}$. In order for both solutions to converge to zero as $t \to \infty$, both coefficients of t in the exponents must be negative.

$$\alpha - 1 < 0$$
$$\alpha < 1$$

Therefore, $y \to 0$ as $t \to \infty$ if $\alpha < 1$. On the other hand, for both solutions to diverge to ∞ as $t \to \infty$, both coefficients of t in the exponents must be positive. Since the coefficient of t in the first solution is fixed at -2, it's not possible for both solutions to diverge.