Problem 26

Consider the initial value problem (see Example 5)

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.

(c) Determine the smallest value of β for which $y_m \geq 4$.

(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

Solution

Part (a)

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 5(re^{rt}) + 6(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 5r + 6 = 0$$
$$(r+3)(r+2) = 0$$
$$r = \{-3, -2\}$$

Two solutions to the ODE are $y = e^{-3t}$ and $y = e^{-2t}$. Therefore, the general solution is

$$y(t) = C_1 e^{-3t} + C_2 e^{-2t},$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = -3C_1 e^{-3t} - 2C_2 e^{-2t}$$

Apply the two initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 2$$
$$y'(0) = -3C_1 - 2C_2 = \beta$$

Solving this system of equations yields $C_1 = -4 - \beta$ and $C_2 = 6 + \beta$. Therefore,

$$y(t) = (-4 - \beta)e^{-3t} + (6 + \beta)e^{-2t}.$$

Part (b)

To find the maximum, solve $y'(t) = 0$ for t and call the result t_m.

$$y'(t) = 3(4+\beta)e^{-3t} - 2(6+\beta)e^{-2t} = 0$$

Multiply both sides by e^{3t}.

$$3(4+\beta) - 2(6+\beta)e^{t} = 0$$
$$2(6+\beta)e^{t} = 3(4+\beta)$$
$$e^{t} = \frac{3(4+\beta)}{2(6+\beta)}$$

Therefore,

$$t_m = t = \ln\frac{3(4+\beta)}{2(6+\beta)}$$

and

$$y_m = y(t_m) = \frac{4(6+\beta)^3}{27(4+\beta)^2}.$$

Part (c)

$$y_m \geq 4$$
$$\frac{4(6+\beta)^3}{27(4+\beta)^2} \geq 4$$
$$\frac{(6+\beta)^3}{(4+\beta)^2} \geq 27$$
$$(6+\beta)^3 \geq 27(4+\beta)^2$$
$$216 + 108\beta + 18\beta^2 + \beta^3 \geq 432 + 216\beta + 27\beta^2$$
$$\beta^3 - 9\beta^2 - 108\beta - 216 \geq 0$$
$$(\beta+3)(\beta^2 - 12\beta - 72) \geq 0$$
$$(\beta+3)[\beta - 6(1-\sqrt{3})][\beta - 6(1+\sqrt{3})] \geq 0$$

Since $\beta > 0$, this inequality is satisfied if β is $6(1+\sqrt{3})$ at a minimum.

Part (d)

$$\begin{aligned}
\lim_{\beta \to \infty} t_m &= \lim_{\beta \to \infty} \ln \frac{3(4+\beta)}{2(6+\beta)} \\
&= \lim_{\beta \to \infty} \ln \frac{3\left(\frac{4}{\beta}+1\right)}{2\left(\frac{6}{\beta}+1\right)} \\
&= \ln \frac{3(0+1)}{2(0+1)} \\
&= \ln \frac{3}{2} \approx 0.405
\end{aligned}$$

$$\begin{aligned}
\lim_{\beta \to \infty} y_m &= \lim_{\beta \to \infty} \frac{4(6+\beta)^3}{27(4+\beta)^2} \\
&= \lim_{\beta \to \infty} \frac{4\beta^3\left(\frac{6}{\beta}+1\right)^3}{27\beta^2\left(\frac{4}{\beta}+1\right)^2} \\
&= \lim_{\beta \to \infty} \frac{4\beta(0+1)^3}{27(0+1)^2} \\
&= \lim_{\beta \to \infty} \frac{4}{27}\beta \\
&= \infty
\end{aligned}$$