Problem 28

Consider the equation $ay'' + by' + cy = 0$, where a, b, and c are constants with $a > 0$. Find conditions on a, b, and c such that the roots of the characteristic equation are:

(a) real, different, and negative.

(b) real with opposite signs.

(c) real, different, and positive.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Part (a)

For the roots to be real, different and negative, $b^2 - 4ac > 0$ and $b > 0$ and $c > 0$.

Part (b)

For the roots to be real with opposite signs, $c < 0$.

Part (c)

For the roots to be real, different and positive, $b^2 - 4ac > 0$ and $b < 0$ and $c > 0$.